REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 68, No. 1, 2025, Pages 231–242
Published online: May 21, 2025
https://doi.org/10.33044/revuma.3886

COMBINATORIAL FORMULAS FOR DETERMINANT, PERMANENT, AND INVERSE OF SOME CIRCULANT MATRICES WITH THREE PARAMETERS

CRISTIAN PANELO, ANDRÉS M. ENCINAS, AND DENIS E. VIDELA

Abstract. We give closed formulas for determinant, permanent, and inverse of circulant matrices with three non-zero coefficients. The techniques that we use are related to digraphs associated with these matrices.

1. Introduction

Circulant matrices appear in many applications, for example, to approximate finite difference of elliptic equations with periodic boundary conditions, or to approximate periodic functions with splines. They play an important role in coding theory and in statistics — the standard reference is [6].

Among the main problems about circulant matrices are determining invertibility conditions and computing their inverse. These problems have been widely treated in the literature by using the primitive n-th root of unity and some polynomial associated with the circulant matrices, see [4, 6, 13]. There exist some classical and well-known results that enable us to solve almost everything we could raise about the inverse of circulant matrices. Nevertheless, when we deal with specific families of circulant matrices, these classical results give us unmanageable formulas. Therefore, it is interesting to find alternative descriptions and in fact, there exist many papers devoted to this question. The direct computation for the inverse of some circulant matrices has been proposed in many works, see [5, 4, 9, 10, 12, 13, 15].

In this work, we delve into the combinatorial structure of circulant matrices with only three non-zero generators, by considering the digraphs associated with this kind of matrices. Therefore, we extend the previous work of some of the authors, see [7, 8], where only an specific class of these matrices was considered.

2020 *Mathematics Subject Classification.* Primary 05C38; Secondary 15A09.

Key words and phrases. circulant matrices, determinant, permanent, inverse, directed weighted cycles.

This work was partially supported by Universidad Nacional de San Luis, Argentina, grant PROICO 03-0918; MATH AmSud, grant 21-MATH-05; and Agencia Nacional de Promoción de la Investigación, el Desarrollo Tecnológico y la Innovación, Argentina, grant PICT 2020-Serie A-00549.

In the present work, we use digraphs. For all of graph-theoretic notions not explicitly defined here, the reader is referred to [1]. As is natural for circulant matrices, our matrix indices and permutations start at zero and so permutations in this work are bijections over $\{0, \dots, n-1\}$, and if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

then the set of indices of A is $\{0, 1\}$; in this case we have that $A_{0\,0} = 1$ and $A_{1\,0} = 3$. With $[n]$ we denote the set $\{0, \dots, n-1\}$ instead of $\{1, \dots, n\}$.

Given a permutation α of $[n]$, we denote by P_α the $n \times n$ matrix defined by $(P_\alpha)_{\alpha(j)\,j} = 1$ and 0 otherwise. The matrix P_α is known as the permutation matrix associated with α. It is well known that $P_\alpha^{-1} = P_\alpha^T$ and $P_{\alpha\rho} = P_\alpha P_\rho$. We use the matrix associated with the permutation

$$\tau_n = (n-1 \quad n-2 \, \cdots \, 2 \;\; 1 \;\; 0)$$

many times along this work, so instead of P_{τ_n} we just write P_n. Notice that, for $k \in \mathbb{Z}$, we have that $P_n^0 = I_n$, $P_n^k = P_n^{(k) \bmod n} = P_{\tau_n^k}$, $\det(P_n) = (-1)^{n-1}$, and $\left(P_n^k\right)^{-1} = P_n^{n-k}$. Moreover, $\tau_n^k(i) = (i-k) \bmod n$.

A matrix $C = (c_{i\,j})$ is called *circulant* with parameters $c_0, c_1, \dots, c_{n-1}$ if

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix};$$

in this case we denote C by $\operatorname{Circ}(c_0, \dots, c_{n-1})$. We have that

$$\operatorname{Circ}(c_0, \dots, c_{n-1}) = c_0 I_n + \cdots + c_{n-1} P_n^{n-1}.$$

The numbers $c_0, \dots, c_{n-1}$ are called the parameters of C. Let a, b, c be non-zero complex numbers. We begin this work by studying the following kind of circulant matrices:

$$\operatorname{Circ}(a, b, c, 0, \dots, 0) = aI_n + bP_n + cP_n^2,$$

which we just call circulant matrices with three parameters.

This paper is organized as follows. In Section 2, we find explicit formulas for the determinant and the permanent of circulant matrices with three parameters. In Section 3, we give an explicit formula for the inverse of non-singular circulant matrices with three parameters. In both sections, we extend the results to a more general case where the parameters are in other positions.

2. Determinant and permanent of the matrices $\mathrm{Circ}(a, b, c, 0, \dots, 0)$

In order to obtain an explicit formula for the determinant of circulant matrices of the form $aI_n + bP_n + cP_n^2$, we have the following definitions.

Definition 2.1 ([3])**.** Let $A = [a_{i\,j}]$ be a matrix of order n. We associate with A a digraph $D(A)$ with n vertices. The vertices of $D(A)$ are denoted by $0, 1, 2, \dots, n-1$. If $a_{i\,j} \neq 0$, there is an arc from vertex i to vertex j of weight $a_{i\,j}$ for each $i, j \in [n]$; we denote this arc by $a_{i\,j}$-arc.

Definition 2.2 ([3])**.** Let D be a digraph. A linear subdigraph of D is a spanning subdigraph of D in which each vertex has indegree 1 and outdegree 1, i.e. exactly one arc into each vertex and exactly one (possibly the same) out of each vertex.

The following theorem gives the determinant and permanent of a given matrix in terms of its associated digraph.

Theorem 2.3 ([3])**.** *Let $A = [a_{i\,j}]$ be a square matrix of order n. Then*

$$\det(A) = \sum_{L \,\in\, \mathcal{L}(D(A))} (-1)^{n-c(L)} w(L)$$

and

$$\mathrm{perm}(A) = \sum_{L \,\in\, \mathcal{L}(D(A))} w(L),$$

where $\mathcal{L}\left(D\left(A\right)\right)$ is the set of all linear subdigraphs of $D(A)$, $c\left(L\right)$ is the number of cycles contained in L (included loops), and $w\left(L\right)$ is the product of the weights of the arcs of L.

Now we study the structure of the digraph $D\left(aI_n + bP_n + cP_n^2\right)$. Since this digraph appears many times in our work, we just write $D_n(a, b, c)$ instead of $D\left(aI_n + bP_n + cP_n^2\right)$.

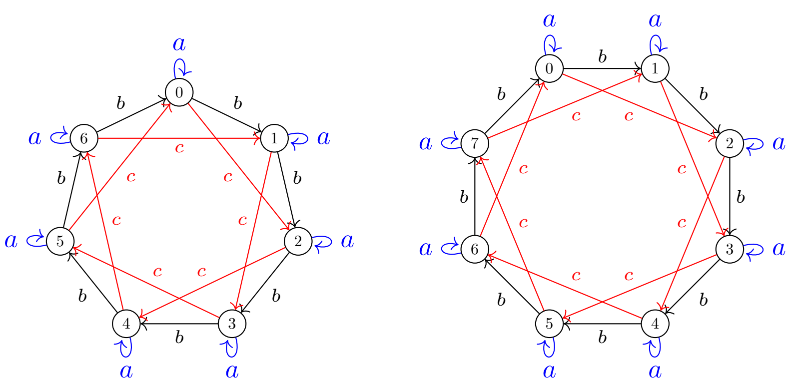
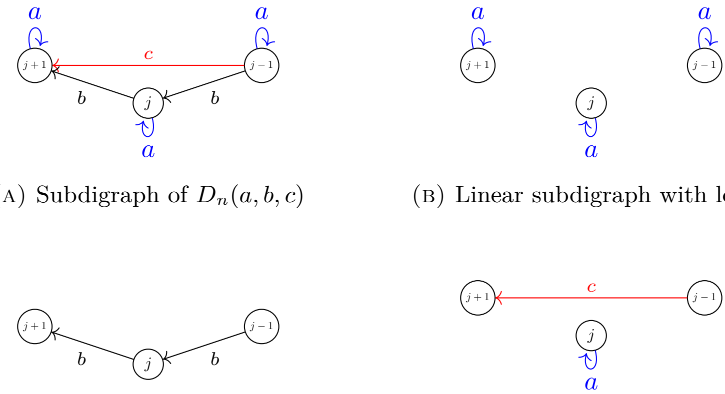


Figure 1. $D_7(a, b, c)$ and $D_8(a, b, c)$

Notice that the digraph $D_n(a, b, c)$ has a directed cycle C, with $V(C) = \{v_0, v_1, \dots, v_{n-1}\}$ such that, for $0 \leq i \leq n-1$, $v_i = i$ and C has a b-arc from v_i to v_{i+1} (with



(A) Subdigraph of $D_n(a,b,c)$

(B) Linear subdigraph with loops

(C) Linear subdigraph with b-arcs

(D) Linear subdigraph with a loop at vertex j

FIGURE 2. Subgraphs of $D_n(a,b,c)$

$v_n = v_0$). From [11], we have that the number of ways to choose k non-adjacent vertices of C, with $k \in [n]$, is

$$\frac{n}{n-k}\binom{n-k}{k}. \tag{2.1}$$

Theorem 2.4. *Let $n \in \mathbb{Z}$ such that $n > 2$ and let $a,b,c \in \mathbb{C} - \{0\}$. Then* $\det\left(aI_n + bP_n + cP_n^2\right)$ *equals*

$$a^n - (-b)^n + c^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-j-1} \frac{n}{n-j}\binom{n-j}{j} a^j b^{n-2j} c^j.$$

Proof. Notice that in the digraph $D_n(a,b,c)$ we have the following linear subdigraphs:

(1) A linear subdigraph L_1 such that L_1 has all the loops (a-arcs). Note that $(-1)^{n-c(L_1)}w(L_1) = (-1)^0 a^n$, see Figure 2b.
(2) A linear subdigraph L_2 such that L_2 is a cycle with b-arcs. Note that $(-1)^{n-c(L_2)}w(L_2) = (-1)^{n-1} b^n$, see Figure 2c.
(3) If n is even, then we have a linear subdigraph L_3 such that L_3 has two cycles of c-arcs. Note that $(-1)^{n-c(L_3)}w(L_3) = (-1)^{n-2}c^n$. If n is odd, then the linear subdigraph L_3 has one cycle of c-arcs. Note that $(-1)^{n-c(L_3)}w(L_3) = (-1)^{n-1}c^n$, see Figure 1.

The other linear subdigraphs have k loops, with $0 < k < n$. Notice that there does not exist a linear subdigraph L with two loops at successive vertices, since if L has loops at j and $j+1$, then L has no arcs from vertex $j-1$. Thus, if we choose the loop at vertex j, then we cannot choose the loops at vertices $j-1$ and $j+1$. By (2.1), we have $\frac{n}{n-k}\binom{n-k}{k}$ choices to take k loops; each of these options gives us a linear subdigraph L_k with k loops such that $(-1)^{n-c(L_k)}w(L_k) = (-1)^{n-k-1}a^k b^{n-2k} c^k$, since for each loop at a vertex j chosen, we lose two b-arcs and we have a c-arc from $j-1$ to $j+1$, see Figure 2d. Moreover, if we choose k loops, then $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, since if we take $k > \lfloor \frac{n}{2} \rfloor$ loops, then the linear subdigraphs is repeated. □

The following corollary follows from Theorems 2.3 and 2.4.

Corollary 2.5. *Let n be a positive integer such that $n > 2$ and let a, b, c be non-zero complex numbers. Then,*

$$\operatorname{perm}\left(aI_n + bP_n + cP_n^2\right) = a^n + b^n + c^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} a^j b^{n-2j} c^j.$$

Example 2.6. Let us consider $A = aI_7 + bP_7 + cP_7^2$ and $B = aI_8 + bP_8 + cP_8^2$. Then

$$\begin{aligned} \det(A) &= a^7 + b^7 + c^7 - 7ab^5c + 14a^2b^3c^2 - 7a^3bc^3, \\ \operatorname{perm}(A) &= a^7 + b^7 + c^7 + 7ab^5c + 14a^2b^3c^2 + 7a^3bc^3, \end{aligned}$$

and

$$\begin{aligned} \det(B) &= a^8 - b^8 + c^8 + 8ab^6c - 20a^2b^4c^2 + 16a^3b^2c^3 - 2a^4c^4, \\ \operatorname{perm}(B) &= a^8 + b^8 + c^8 + 8ab^6c + 20a^2b^4c^2 + 16a^3b^2c^3 + 2a^4c^4. \end{aligned}$$

With $\otimes$ we denote the usual Kronecker product between matrices, see [14]. Following the notation in [8], given n and s two non-zero integers we have the number $n \backslash s := \frac{n}{\gcd(n,s)}$.

Theorem 2.7 ([8])**.** *Let $n, s \in \mathbb{Z}$ such that $0 < s < n$ and let $a_k \in \mathbb{C} - \{0\}$ for $k \in [n \backslash s]$. Then, there exists a permutation $\sigma_{n,s}$ of $[n]$ such that*

$$P_{\sigma_{n,s}}^T \left(\sum_{k=0}^{(n\backslash s)-1} a_k P_n^{sk} \right) P_{\sigma_{n,s}} = I_{\gcd(n,s)} \otimes \left(\sum_{k=0}^{(n\backslash s)-1} a_k P_{n\backslash s}^k \right).$$

By Theorem 2.7 we have the following corollary.

Corollary 2.8. *Let $n, s_1, s_2, s_3 \in \mathbb{Z}$ such that $0 \leq s_1 < s_2 < s_3 \leq n$, and let $a, b, c \in \mathbb{C} - \{0\}$. If $s_3 - s_1 = 2\,(s_2 - s_1)$, then*

$$aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3} = P_n^{s_1} P_{\sigma_{n,s}} \left(I_{\gcd(n,s)} \otimes \left(aI_{n\backslash s} + bP_{n\backslash s} + cP_{n\backslash s}^2 \right) \right) P_{\sigma_{n,s}}^T,$$

where $s = s_2 - s_1$.

By Corollary 2.8 we have the following corollary.

Corollary 2.9. *Let $n, s_1, s_2, s_3 \in \mathbb{Z}$ such that $0 \leq s_1 < s_2 < s_3 \leq n$, and let $a, b, c \in \mathbb{C} - \{0\}$. If $s_3 - s_1 = 2(s_2 - s_1)$, then $\det(aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3})$ equals*

$$(-1)^{s_1(n-1)}\left(\det\left(aI_{n\backslash(s_2-s_1)}+bP_{n\backslash(s_2-s_1)}+cP^2_{n\backslash(s_2-s_1)}\right)\right)^{\gcd(n,s_2-s_1)}.$$

Proof. Let $s = s_2 - s_1$ and $A = aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3}$. By Corollary 2.8,

$$\begin{aligned}\det(A) &= \det\left(P_n^{s_1}P_{\sigma_{n,s}}\left(I_{\gcd(n,s)}\otimes\left(aI_{n\backslash s}+bP_{n\backslash s}+cP^2_{n\backslash s}\right)\right)P^T_{\sigma_{n,s}}\right)\\ &= \det\left(P_n^{s_1}\right)\det\left(I_{\gcd(n,s)}\otimes\left(aI_{n\backslash s}+bP_{n\backslash s}+cP^2_{n\backslash s}\right)\right).\end{aligned}$$

Let C and D be matrices of order n and m respectively; then $\det(C \otimes D) = (\det(C))^m(\det(D))^n$, see [14]. On the other hand, $\det(P_n^{s_1}) = (-1)^{s_1(n-1)}$, see [8]. Therefore,

$$\det(A) = (-1)^{s_1(n-1)}\left(\det\left(aI_{n\backslash s}+bP_{n\backslash s}+cP^2_{n\backslash s}\right)\right)^{\gcd(n,s)}. \qquad \square$$

Given a square matrix A of order n. We have that $\operatorname{perm}(PA) = \operatorname{perm}(A)$, where P is a permutation matrix or order n, and $\operatorname{perm}(I_m \otimes A) = (\operatorname{perm}(A))^m$, see [2]. Thus, we have the following corollary.

Corollary 2.10. *Let $n, s_1, s_2, s_3 \in \mathbb{Z}$ such that $0 \leq s_1 < s_2 < s_3 \leq n-1$, and let $a, b, c \in \mathbb{C} - \{0\}$. If $s_3 - s_1 = 2(s_2 - s_1)$, then $\operatorname{perm}(aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3})$ equals*

$$\left(\operatorname{perm}\left(aI_{n\backslash(s_2-s_1)}+bP_{n\backslash(s_2-s_1)}+cP^2_{n\backslash(s_2-s_1)}\right)\right)^{\gcd(n,s_2-s_1)}.$$

3. Inverse of the matrices $\operatorname{Circ}(a, b, c, 0, \ldots, 0)$

For the rest of the work we use the following convention: let $j, k, t \in \mathbb{Z}$, and let $x_j \in \mathbb{C} - \{0\}$. We set $\sum_{j=t}^{k} x_j = 0$ and $\binom{k}{t} = 0$ when $k < t$. Also, we set $y^\ell = z^\ell = 0$ when $\ell < 0$, for $y, z \in \mathbb{C}$ and $\ell \in \mathbb{R}$.

Definition 3.1. Let $i, n \in \mathbb{Z}$ such that $n > 0$, and let $a, b, c \in \mathbb{C} - \{0\}$. We define

$$\begin{aligned}L_n(i) = {}&(-1)^t a^{n-1-t}b^t + (-1)^{n-2-t}b^{n-2-t}c^{t+1}\\ &+\sum_{j=1}^{\lfloor\frac{n-2-t}{2}\rfloor}(-1)^{n-t-j}\binom{n-2-t-j}{j}a^jb^{n-2-t-2j}c^{t+j+1}\\ &+\sum_{j=1}^{\lfloor\frac{t}{2}\rfloor}(-1)^{t-j}\binom{t-j}{j}a^{n-1-t+j}b^{t-2j}c^j,\end{aligned}$$

where $t = (i) \bmod n$.

Example 3.2. The table shows the values of $L_n(i)$ for $n=7$ and $n=8$.

i	$L_7(i)$	i	$L_8(i)$
0	$a^6 - b^5c + 4ab^3c^2 - 3a^2bc^3$	0	$a^7 + b^6c - 5ab^4c^2 + 6a^2b^2c^3 - a^3c^4$
1	$-a^5b + b^4c^2 - 3ab^2c^3 + a^2c^4$	1	$-a^6b - b^5c^2 + 4ab^3c^3 - 3a^2bc^4$
2	$a^4b^2 - b^3c^3 + 2abc^4 - a^5c$	2	$a^5b^2 + b^4c^3 - 3ab^2c^4 + a^2c^5 - a^6c$
3	$-a^3b^3 + b^2c^4 - ac^5 + 2a^4bc$	3	$-a^4b^3 - b^3c^4 + 2abc^5 + 2a^5bc$
4	$a^2b^4 - bc^5 - 3a^3b^2c + a^4c^2$	4	$a^3b^4 + b^2c^5 - ac^6 - 3a^4b^2c + a^5c^2$
5	$-ab^5 + c^6 + 4a^2b^3c - 3a^3bc^2$	5	$-a^2b^5 - bc^6 + 4a^3b^3c - 3a^4bc^2$
6	$b^6 - 5ab^4c + 6a^2b^2c^2 - a^3c^3$	6	$ab^6 + c^7 - 5a^2b^4c + 3a^3b^2c^2 - a^4c^3$
		7	$-b^7 + 6ab^5c - 10a^2b^3c^2 + 4a^3bc^3$

Notice that when $i,n,t \in \mathbb{Z}$ are such that $n>2$, $1 \le i \le n-1$ and $1 \le t \le \left\lfloor \frac{n-i}{2} \right\rfloor$, we have the following useful properties that can be proved by simple calculations:

$$\left.\begin{array}{r}\binom{n-1-i-t}{t-1} + \binom{n-1-i-t}{t} = \binom{n-i-t}{t} \\ \binom{i-1-t}{t-1} + \binom{i-1-t}{t} = \binom{i-t}{t},\end{array}\right\} \tag{3.1}$$

$$2\binom{n-1-t}{t-1} + \binom{n-1-t}{t} = \frac{n}{n-t}\binom{n-t}{t}, \tag{3.2}$$

$$\left\lfloor \frac{n-i}{2} \right\rfloor = \left\lfloor \frac{n-2-i}{2} \right\rfloor + 1,$$

and

$$\left\lfloor \frac{n-i}{2} \right\rfloor = \begin{cases} \left\lfloor \frac{n-1-i}{2} \right\rfloor & \text{if } n \neq (i) \bmod 2, \\ \left\lfloor \frac{n-1-i}{2} \right\rfloor + 1 & \text{if } n = (i) \bmod 2. \end{cases} \tag{3.3}$$

In order to obtain an explicit formula for the inverse of a non-singular matrix of the form $aI_n + bP_n + cP_n^2$, we define

$$\rho_n(a,b,c,i) := a\,L_n(i) + b\,L_n(i-1) + c\,L_n(i-2).$$

Lemma 3.3. *Let $i,n \in \mathbb{Z}$ such that $n>2$, and let $a,b,c \in \mathbb{C}-\{0\}$. If $1 \le i \le n-1$, then $\rho_n(a,b,c,i) = 0$.*

Proof. Let $i > 0$. Let us consider

$$x_{a,i} = (-1)^{n-2-i} a b^{n-2-i} c^{i+1} + \sum_{j=1}^{\left\lfloor \frac{n-2-i}{2} \right\rfloor} \alpha_{i,j} + \sum_{j=1}^{\left\lfloor \frac{n-1-i}{2} \right\rfloor} \beta_{i,j} + \sum_{j=1}^{\left\lfloor \frac{n-i}{2} \right\rfloor} \gamma_{i,j}$$

and

$$x_{c,i} = (-1)^{i-2} a^{n+1-i} b^{i-2} c + \sum_{j=1}^{\left\lfloor \frac{i-2}{2} \right\rfloor} \alpha'_{i,j} + \sum_{j=1}^{\left\lfloor \frac{i}{2} \right\rfloor} \beta'_{i,j} + \sum_{j=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor} \gamma'_{i,j},$$

where

$$\begin{aligned} \alpha_{i,j} &= (-1)^{n-i-j} \binom{n-2-i-j}{j} a^{j+1} b^{n-2-i-2j} c^{i+j+1}, \\ \beta_{i,j} &= (-1)^{n+1-i-j} \binom{n-1-i-j}{j} a^{j} b^{n-i-2j} c^{i+j}, \\ \gamma_{i,j} &= (-1)^{n+2-i-j} \binom{n-i-j}{j} a^{j} b^{n-i-2j} c^{i+j}, \end{aligned}$$

and

$$\begin{aligned} \alpha'_{i,j} &= (-1)^{i-2-j} \binom{i-2-j}{j} a^{n+1-i+j} b^{i-2-2j} c^{j+1}, \\ \beta'_{i,j} &= (-1)^{i-j} \binom{i-j}{j} a^{n-i+j} b^{i-2j} c^{j}, \\ \gamma'_{i,j} &= (-1)^{i-1-j} \binom{i-1-j}{j} a^{n-i+j} b^{i-2j} c^{j}. \end{aligned}$$

Claim: $x_{a,i} = x_{c,i} = 0$.

Note that if we consider $x_{a,i}$ as a polynomial on a, b, c, we have that $x_{a,i}$ only has monomials of the form $a^j b^{n-i-2j} c^{i+j}$, with $1 \leq j \leq \left\lfloor \frac{n-i}{2} \right\rfloor$, i.e.

$$x_{a,i} = \sum_{j=1}^{\left\lfloor \frac{n-i}{2} \right\rfloor} A_j \, a^j b^{n-i-2j} c^{i+j},$$

where, by (3.3), we have that

$$A_1 = (-1)^{n-2-i} a b^{n-2-i} c^{i+1} + \beta_{i,1} + \gamma_{i,1}$$

and, if $i \neq (n) \bmod 2$, then for $2 \leq j \leq \left\lfloor \frac{n-i}{2} \right\rfloor$

$$A_j = \alpha_{i,j-1} + \beta_{i,j} + \gamma_{i,j},$$

and if $i = (n) \bmod 2$, then

$$A_j = \begin{cases} \alpha_{i,j-1} + \beta_{i,j} + \gamma_{i,j} & \text{if } 2 \leq j \leq \left\lfloor \frac{n-i}{2} \right\rfloor - 1, \\ \alpha_{i,j-1} + \gamma_{i,j} & \text{if } j = \left\lfloor \frac{n-i}{2} \right\rfloor. \end{cases}$$

By (3.1) we have that $A_j = 0$ for all j. Therefore, $x_{a,i} = 0$ as desired.

A similar argument shows that $x_{c,i} = 0$. Therefore, $x_{a,i} = x_{c,i} = 0$, as we claimed.

Now, notice that

$$a \, L_n(i) = (-1)^i a^{n-i} b^i + (-1)^{n-2-i} a b^{n-2-i} c^{i+1}$$

$$+\sum_{j=1}^{\lfloor\frac{n-2-i}{2}\rfloor}(-1)^{n-i-j}\binom{n-2-i-j}{j}a^{j+1}b^{n-2-i-2j}c^{i+j+1}$$
$$+\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor}(-1)^{i-j}\binom{i-j}{j}a^{n-i+j}b^{i-2j}c^{j},$$

$$\begin{aligned} b\,L_n(i-1) &= (-1)^{i-1}a^{n-i}b^{i}+(-1)^{n-1-i}b^{n-i}c^{i}\\ &\quad+\sum_{j=1}^{\lfloor\frac{n-1-i}{2}\rfloor}(-1)^{n+1-i-j}\binom{n-1-i-j}{j}a^{j}b^{n-i-2j}c^{i+j}\\ &\quad+\sum_{j=1}^{\lfloor\frac{i-1}{2}\rfloor}(-1)^{i-1-j}\binom{i-1-j}{j}a^{n-i+j}b^{i-2j}c^{j}\end{aligned}$$

and

$$\begin{aligned} c\,L_n(i-2) &= (-1)^{i-2}a^{n+1-i}b^{i-2}c+(-1)^{n-i}b^{n-i}c^{i}\\ &\quad+\sum_{j=1}^{\lfloor\frac{n-i}{2}\rfloor}(-1)^{n+2-i-j}\binom{n-i-j}{j}a^{j}b^{n-i-2j}c^{i+j}\\ &\quad+\sum_{j=1}^{\lfloor\frac{i-2}{2}\rfloor}(-1)^{i-2-j}\binom{i-2-j}{j}a^{n+1-i+j}b^{i-2-2j}c^{j+1}.\end{aligned}$$

For $1\le i\le n-1$, we have that

$$a\,L_n(i)+b\,L_n(i-1)+c\,L(i-2)=x_{a,i}+x_{c,i}.$$

Since $x_{a,i}=x_{c,i}=0$, we obtain $\rho_n(a,b,c,i)=0$, as asserted. □

Lemma 3.4. *Let $n\in\mathbb{Z}$ such that $n>2$, and let $a,b,c\in\mathbb{C}-\{0\}$. Then*

$$\rho_n(a,b,c,0)=\det\left(aI_n+bP_n+cP_n^2\right).$$

Proof. We have that

$$\begin{aligned}\rho_n(a,b,c,0)&=a\,L_n(0)+b\,L_n(-1)+c\,L_n(-2)\\ &=a\,L_n(0)+b\,L_n(n-1)+c\,L_n(n-2)\end{aligned}$$

and

$$\begin{aligned} L_n(0) &= a^{n-1}+(-1)^{n-2}b^{n-2}c\\ &\quad+\sum_{j=1}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^{n-j}\binom{n-2-j}{j}a^{j}b^{n-2-2j}c^{j+1}\\ &\quad+\sum_{j=1}^{0}(-1)^{-j}\binom{-j}{j}a^{n-1+j}b^{-2j}c^{j},\end{aligned}$$

$$
\begin{aligned}
L_n(n-1) = (-1)^{n-1}b^{n-1} & \\
&+ \sum_{j=1}^{-1} (-1)^{1-j}\tbinom{-1-j}{j} a^j b^{-1-2j} c^{n+j} \\
&+ \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-1-j}\tbinom{n-1-j}{j} a^j b^{n-1-2j} c^j, \\
L_n(n-2) = (-1)^{n-2}ab^{n-2} + c^{n-1} & \\
&+ \sum_{j=1}^{0} (-1)^{2-j}\tbinom{-j}{j} a^j b^{-2j} c^{n-1+j} \\
&+ \sum_{j=1}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^{n-2-j}\tbinom{n-2-j}{j} a^{1+j} b^{n-2-2j} c^j.
\end{aligned}
$$

Then $a\,L_n(0) + b\,L_n(n-1) + c\,L_n(n-2)$ is equal to

$$
a^n - (-b)^n + c^n + (-1)^{n-2} 2\,ab^{n-2}c + \sum_{j=1}^{\lfloor \frac{n-2}{2} \rfloor} \alpha_{0,j} + \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \beta_{0,j},
$$

where

$$
\begin{aligned}
\alpha_{0,j} &= (-1)^{n-j} 2 \tbinom{n-2-j}{j} a^{j+1} b^{n-2-2j} c^{j+1}, \quad \text{and} \\
\beta_{0,j} &= (-1)^{n-1-j} \tbinom{n-1-j}{j} a^j b^{n-2j} c^j.
\end{aligned}
$$

Note that if we consider $\rho_n(a,b,c,0) - a^n + (-b)^n - c^n$ as a polynomial on a,b,c we have that it only has monomials of the form $a^j b^{n-j} c^j$, i.e.

$$
\rho_n(a,b,c,0) - a^n + (-b)^n - c^n = \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} B_j\, a^j b^{n-2j} c^j,
$$

where, by (3.3), we have that

$$
B_1 = (-1)^{n-2} 2\, ab^{n-2}c + \beta_{0,1}
$$

and, if n is odd, then for $2 \le j \le \lfloor \frac{n}{2} \rfloor$

$$
B_j = \alpha_{0,j-1} + \beta_{0,j},
$$

and, if n is even, then

$$
B_j = \begin{cases} \alpha_{0,j-1} + \beta_{0,j} & \text{if } 2 \le j \le \lfloor \frac{n}{2} \rfloor - 1, \\ \alpha_{0,j-1} & \text{if } j = \lfloor \frac{n}{2} \rfloor. \end{cases}
$$

By (3.2) we have that $B_j = (-1)^{n-j-1} \frac{n}{n-j} \binom{n-j}{j} a^j b^{n-2j} c^j$ for all j. Therefore, $\rho_n(a,b,c,0) = \det\left(aI_n + bP_n + cP_n^2\right)$, as desired. □

Theorem 3.5. *Let $n \in \mathbb{Z}$ such that $n > 2$, and let $a, b, c \in \mathbb{C} - \{0\}$. Then*

$$\left(aI_n + bP_n + cP_n^2\right)^{-1} = \frac{1}{\det\left(aI_n + bP_n + cP_n^2\right)} \sum_{i=0}^{n-1} L_n(i) P_n^i.$$

Proof. Follows directly from Lemmas 3.3 and 3.4, since

$$\left(aI_n + bP_n + cP_n^2\right)\left(\sum_{i=0}^{n-1} \frac{L_n(i)}{m} P_n^i\right) = \sum_{i=0}^{n-1} \frac{\rho_n(i)}{m} P_n^i,$$

where $m = \det\left(aI_n + bP_n + cP_n^2\right)$. □

Finally, by Corollary 2.8 and Theorem 3.5 we have the following theorem.

Theorem 3.6. *Let $n, s_1, s_2, s_3 \in \mathbb{Z}$ such that $0 \le s_1 < s_2 < s_3 \le n-1$, and let $a, b, c \in \mathbb{C} - \{0\}$. If $s_3 - s_1 = 2s$, where $s = s_2 - s_1$, and the matrix $aI_{n\backslash s} + bP_{n\backslash s} + cP_{n\backslash s}^2$ is non-singular, then $aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3}$ is non-singular and its inverse matrix is*

$$P_{\sigma_{n,s}}\left(I_r \otimes \left(\frac{1}{\det\left(aI_t + bP_t + cP_t^2\right)} \sum_{i=0}^{t-1} L_t(i) P_t^i\right)\right) P_{\sigma_{n,s}}^T P_n^{n-s_1},$$

where $r = \gcd(n, s)$ and $t = n\backslash s$.

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, algorithms and applications*, second ed., Springer Monographs in Mathematics, Springer, London, 2009. DOI MR Zbl

[2] R. A. Brualdi, Permanent of the direct product of matrices, *Pacific J. Math.* **16** (1966), 471–482. DOI MR Zbl

[3] R. A. Brualdi and D. Cvetković, *A combinatorial approach to matrix theory and its applications*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2009. DOI MR Zbl

[4] A. Carmona, A. M. Encinas, S. Gago, M. J. Jiménez, and M. Mitjana, The inverses of some circulant matrices, *Appl. Math. Comput.* **270** (2015), 785–793. DOI MR Zbl

[5] A. Carmona, A. M. Encinas, M. J. Jimenez, and M. Mitjana, The group inverse of circulant matrices depending on four parameters, *Spec. Matrices* **10** (2022), 87–108. DOI MR Zbl

[6] P. J. Davis, *Circulant matrices*, second ed., AMS Chelsea Publishing, New York, 1994. Zbl

[7] A. M. Encinas, D. A. Jaume, C. Panelo, and A. Pastine, Drazin inverse of singular adjacency matrices of directed weighted cycles, *Rev. Un. Mat. Argentina* **61** no. 2 (2020), 209–227. DOI MR Zbl

[8] A. M. Encinas, D. A. Jaume, C. Panelo, and D. E. Videla, Using digraphs to compute determinant, permanent, and Drazin inverse of circulant matrices with two parameters, *Rev. Un. Mat. Argentina* **67** no. 1 (2024), 81–106. DOI MR Zbl

[9] L. Fuyong, The inverse of circulant matrix, *Appl. Math. Comput.* **217** no. 21 (2011), 8495–8503. DOI MR Zbl

[10] Z. Jiang, Y. Gong, and Y. Gao, Invertibility and explicit inverses of circulant-type matrices with k-Fibonacci and k-Lucas numbers, *Abstr. Appl. Anal.* (2014), article no. 238953. DOI MR Zbl

[11] T. Mansour and Y. Sun, On the number of combinations without certain separations, *European J. Combin.* **29** no. 5 (2008), 1200–1206. DOI MR Zbl

[12] B. Radičić, On k-circulant matrices (with geometric sequence), *Quaest. Math.* **39** no. 1 (2016), 135–144. DOI MR Zbl

[13] S.-Q. Shen, J.-M. Cen, and Y. Hao, On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers, *Appl. Math. Comput.* **217** no. 23 (2011), 9790–9797. DOI MR Zbl

[14] W.-H. Steeb and Y. Hardy, *Matrix calculus and Kronecker product*, second ed., World Scientific, Hackensack, NJ, 2011. DOI MR Zbl

[15] Y. Yazlik and N. Taskara, On the inverse of circulant matrix via generalized k-Horadam numbers, *Appl. Math. Comput.* **223** (2013), 191–196. DOI MR Zbl

Cristian Panelo ✉
Departamento de Matemáticas, Facultad de Ciencias Físico-Matemáticas y Naturales,
Universidad Nacional de San Luis, San Luis, Argentina
crpanelo@unsl.edu.ar

Andrés M. Encinas
Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Spain
andres.marcos.encinas@upc.edu

Denis E. Videla
FaMAF – CIEM (CONICET), Universidad Nacional de Córdoba, Córdoba, Argentina
devidela@famaf.unc.edu.ar

Received: January 31, 2023
Accepted: June 27, 2023
Early view: August 24, 2024