REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 64, No. 2, 2023, Pages 281–308
Published online: February 27, 2023
https://doi.org/10.33044/revuma.2918

NORMAL FORM TRANSFORMATIONS FOR MODULATED DEEP-WATER GRAVITY WAVES

PHILIPPE GUYENNE, ADILBEK KAIRZHAN, AND CATHERINE SULEM

ABSTRACT. Modulation theory is a well-known tool to describe the long-time evolution and stability of small-amplitude, oscillating solutions to dispersive nonlinear partial differential equations. There have been a number of approaches to deriving envelope equations for weakly nonlinear waves. Here we review a systematic method based on Hamiltonian transformation theory and averaging Hamiltonians. In the context of the modulation of two- or three-dimensional deep-water surface waves, this approach leads to a Dysthe equation that preserves the Hamiltonian character of the water wave problem. An explicit calculation of the third-order Birkhoff normal form that eliminates all non-resonant cubic terms yields a non-perturbative procedure for the reconstruction of the free surface. We also present new numerical simulations of this weakly nonlinear approximation using the version with exact linear dispersion. We compare them against computations from the full water wave system and find very good agreement.

1. Introduction

The equations for water waves describe the flow of an incompressible and irrotational fluid with a free surface, under the restoring forces of gravity. It has been known since the seminal paper of Zakharov [41] that this system of equations can be written in the form of a Hamiltonian system with Hamiltonian

$$H(\eta,\xi) = \frac{1}{2}\int (\xi G(\eta)\xi + g\eta^2)\,\mathrm{d}x,$$

with a standard Darboux symplectic structure. Canonical conjugate variables are given by (η,ξ), where $\eta(x,t)$ is the surface elevation and $\xi(x,t) = \varphi(x,\eta(x,t),t)$ is the boundary value of the velocity potential on the free surface, and $G(\eta)$ is the Dirichlet–Neumann operator associated to the fluid domain. The stationary solution $(\eta,\xi) = (0,0)$ corresponding to the fluid at rest is an elliptic stationary point

2020 *Mathematics Subject Classification.* 76B15, 35Q55.

Key words and phrases. Deep-water gravity waves, Dysthe equation, exact linear dispersion, Hamiltonian systems, modulational analysis, normal form transformations, numerical simulations.

A. K. is partially supported by McMaster University. C. S. is partially supported by the Natural Sciences and Engineering Research Council of Canada (grant number 2018-04536) and a Killam Research Fellowship from the Canada Council for the Arts.

in dynamical systems terms. The method of Birkhoff normal form transformations provides a mechanism to eliminate non-essential nonlinearities. In this context, Dyachenko and Zakharov [18] and Craig and Wolfork [15] showed that, in two dimensions with infinite depth, the Hamiltonian has no three-wave resonances, and that the four-wave resonances have a special integrable form. At a formal level, this reduction to fourth-order Birkhoff normal form expresses that cubic terms and non-resonant quartic terms can be removed by appropriate canonical transformations. The remaining resonant quartic terms are expressed in terns of action variables alone. The mapping properties of these transformations were studied in [13].

The integrability of the water wave system truncated at fourth order has been at the centre of recent analytic works and plays an important role in the rigorous proof of long-time behavior of small solutions for the two-dimensional water wave problem. In [2, 3], Berti, Feola and Pusateri consider the two-dimensional gravity water wave system with a periodic one-dimensional interface on infinite depth and provide a rigorous reduction of these equations to Birkhoff normal form up to degree four. For an initial interface in the form of a periodic perturbation of size ε of the flat surface in a Sobolev space of sufficient regularity, they prove that the solution of the water wave problem remains smooth and small up to time of order $O(\varepsilon^{-3})$.

This partial integrability appears through a different point of view in Wu's recent work [39], where she establishes a long-time existence for solutions of the water wave problem in $\mathbb{R}^2$ without imposing any size restrictions on the slope of the initial interface and on the magnitude of the initial velocity. The two-dimensional water wave problem is written in holomorphic coordinates after a conformal map transforms the fluid domain to the lower half-plane. Wu constructs, in the Riemann mapping coordinates and after some transformations, a sequence of energy functionals $E_j(t)$ such that $dE_j(t)/dt$ are quintic or higher order. This can be interpreted as a reflection of the quartic integrability. These energy functionals, $E_j(t)$ for $j \geq 2$, are expressed in terms of derivatives of the velocity and derivatives of the steepness, allowing for solutions that are small in $E_j(t)$, but with large velocity and large steepness. She proves that if the initial conditions are of size ε in a scaling invariant norm, the lifespan of the solutions is at least $O(\varepsilon^{-5/2})$. This condition does not require that the slope of the initial interface and the magnitude of the initial velocity be small.

Modulation theory is a well-established tool to describe envelope equations for weakly nonlinear narrowband wave trains. The present article is based on Hamiltonian transformations and their application to the long-time evolution of slowly modulated, small-amplitude monochromatic waves. By construction, this approach provides a systematic method for the derivation of Hamiltonian amplitude equations.

Due to the presence of quadratic nonlinearities in the water wave system, a mean flow is driven by the amplitude modulation. The slowly varying envelope is described as a solution of the Nonlinear Schrödinger (NLS) equation [41, 27], or more generally, the Davey–Stewartson system in three dimensions [16, 1] (see also [35]). A higher-order approximation for the envelope equation was proposed by Dysthe [20] for deep-water gravity waves using the method of multiple scales, and it was later extended to several other physical settings, for example, finite depth [4], gravity-capillary waves [28] and exact linear dispersion [38] to cite a few. The Dysthe equation and its variants are widely used in the oceanographic community because of their efficiency to capture realistic waves, in particular waves of moderately large steepness. Among many instances, we mention the use of the Dysthe equation to study the stability of wave trains [37] and the formation of rogue waves [36]. It was also observed that numerical solutions of the Dysthe model provide a better agreement with laboratory experiments than NLS and, in particular, capture the asymmetry of wave packets, a feature not captured by NLS [31, 34]. Unlike the cubic NLS model, which is a canonical Hamiltonian partial differential equation (PDE), earlier versions of the Dysthe equation do not preserve the Hamiltonian character of the primitive equations.

Gramstad and Trulsen [22] used a Zakharov's four-wave interaction model as obtained by Krasitskii [30] and derived a Hamiltonian version of Dysthe's equation for three-dimensional gravity surface waves on finite depth.

In [10] and [24] we considered, respectively, the two- and three-dimensional problem of gravity waves on deep water in the modulational regime. We derived one-dimensional and two-dimensional Dysthe equations directly from the two- and three-dimensional Euler equations for potential flow through a sequence of canonical transformations involving various asymptotic scalings, Birkhoff normal forms, a modulational Ansatz as well as a homogenization lemma. By construction, the resulting Dysthe equations preserve the Hamiltonian character of the problem. These approximations were tested against direct numerical simulations of the three-dimensional Euler system and against predictions from the classical Dysthe equation, and we found very good agreement. The above works use in a central way Birkhoff normal forms to eliminate all non-resonant cubic terms, while the modulational Ansatz naturally helps select the resonant quartic terms.

It is to be noted that the quantity that satisfies the Dysthe equation is not exactly the envelope of the free surface. A novelty of our approach is a direct reconstruction of the free surface from the solution of the Dysthe equation. Classically, this reconstruction is performed perturbatively in terms of a Stokes expansion [22]. Our procedure is achieved through a non-perturbative method that requires solving, for the two-dimensional problem, an inviscid Burgers equation associated to the cubic Birkhoff normal form transformation. For the three-dimensional problem, this Burgers equation is replaced by a less simple, still explicit system. This surface-reconstruction step is consistent with the Hamiltonian framework and involves the solution of an auxiliary Hamiltonian system.

In a subsequent paper [25], we derived a spatial form of this Hamiltonian Dysthe equation, better adapted to comparison with measurements along a wave tank in laboratory experiments, and we tested it against the classical results of Lo and Mei [31]. We mention that a Hamiltonian version of the Dysthe equation was derived by Fedele and Dutykh [21] but it is not obtained directly from the water wave problem. Instead, the authors start from a generic form of a high-order NLS equation with arbitrary coefficients in factor of nonlinear terms, and find those that will make this equation Hamiltonian.

We also mention a special reduction of Zakharov's equation for two-dimensional gravity waves on deep water (referred to as the compact or super compact equation) by Dyachenko et al. [19, 17]. Using the quartic integrability of this model, the authors derive a simplified equation that includes a nonlinear term similar to that of NLS as well as an advection term that can describe the initial stage of wave breaking. Finally, normal form transformations play a central role in Zakharov's theory of wave turbulence, where nonlinear wave interactions are reduced to resonant submanifolds under canonical changes of variables [42, 33].

In the present paper, we give a detailed review of our methodology and results. We also show new numerical simulations of the Hamiltonian Dysthe equation with exact linear dispersion in the context of modulational instability of Stokes waves. We compare the resulting predictions to those from the full water wave system and find very good agreement.

We conclude this introduction with a discussion on well-posedness properties of the Dysthe equation, which is a key element in the effort to rigorously prove the validity of the modulational Ansatz. Using techniques developed for NLS with nonlinear terms containing derivatives, Chihara [5] proved local well-posedness of the (classical) Dysthe equation for initial data in $H^3(\mathbb{R}^2)$. It was extended to initial data in $H^s(\mathbb{R}^2)$, $s \geq 3/2$, by Koch and Saut [29]. Further improvements have been obtained in two recent papers. In [23], Grande, Kurianski and Staffilani establish a local well-posedness result for small initial conditions in $H^s(\mathbb{R}^2)$, $s > 1$. They also prove that the initial value problem is ill-posed in $H^s(\mathbb{R}^2)$ for $s < 0$, in the sense that the flow map (data-solution) cannot be C^3 in $H^s(\mathbb{R}^2)$, $s < 0$. Furthermore, Mosincat, Pilod and Saut [32] prove global well-posedness and scattering for small data in the critical space $L^2(\mathbb{R}^2)$. This result is sharp in view of the ill-posedness result cited above.

2. The water wave system and its Hamiltonian formulation

The classical water wave problem refers to the motion of an ideal incompressible fluid in the presence of gravity. The fluid domain

$$S(\eta, t) = \{(x, y) : x \in \mathbb{R}^{d-1},\ -\infty < y < \eta(x, t)\},$$

where $d = 2$ or 3 is the spatial dimension and $y = \eta(x, t)$ represents the free surface at time t on infinite depth. We will mostly report on the case $d = 2$ and refer the reader to a recent article [24] for the extension to the three-dimensional problem. As it is often the case, we assume the fluid is irrotational and thus described by a potential flow $u = \nabla\varphi$ satisfying

$$\Delta\varphi = 0$$

in the fluid domain $S(\eta,t)$. On the interface $\{y=\eta(x,t)\}$, two boundary conditions are imposed, namely

$$\partial_t\eta = \partial_y\varphi - \partial_x\eta\cdot\partial_x\varphi,$$
$$\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta = 0,$$

where g is the acceleration due to gravity and $\nabla = (\partial_x,\partial_y)$.

A remarkable property of the water wave system, discovered by Zakharov [41], is that it has a Hamiltonian formulation with canonical variables $(\eta(x,t),\xi(x,t) := \varphi(x,\eta(x,t),t))$ in the form

$$\partial_t\begin{pmatrix}\eta\\ \xi\end{pmatrix} = J\ \nabla_{\eta,\xi}H(\eta,\xi) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\partial_\eta H\\ \partial_\xi H\end{pmatrix},$$

where the Hamiltonian $H(\eta,\xi)$ is the total energy

$$\begin{aligned}H(\eta,\xi) &= \frac{1}{2}\iint_{-\infty}^{\eta(x,t)}|\nabla\varphi|^2\,\mathrm{d}y\mathrm{d}x + \frac{g}{2}\int\eta^2\,\mathrm{d}x,\\ &= \frac{1}{2}\int(\xi G(\eta)\xi\ + g\eta^2)\,\mathrm{d}x.\end{aligned}$$

It is understood that the integration in x (as well as in the Fourier space) is over $\mathbb{R}^{d-1}$, which is hereafter omitted for convenience. Here $G(\eta)$ is the Dirichlet–Neumann operator (DNO) which associates to the Dirichlet data ξ on $y=\eta(x,t)$ the normal derivative of the harmonic function φ with a normalized factor, namely

$$G(\eta):\xi\longmapsto\sqrt{1+|\partial_x\eta|^2}\,\partial_n\varphi\Big|_{y=\eta}.$$

The other conserved quantities are the mass (or volume)

$$V(\eta)=\int\eta\,\mathrm{d}x,$$

and the horizontal momentum (or impulse)

$$I(\eta,\xi) = -\int\xi\partial_x\eta\,\mathrm{d}x. \tag{2.1}$$

Defining the Poisson bracket as

$$\{F,G\} = \int(\partial_\eta F\partial_\xi G - \partial_\xi F\partial_\eta G)\,\mathrm{d}x,$$

one checks that the conserved quantities V and I Poisson-commute with the Hamiltonian H:

$$\{H,V\}=0,\qquad \{H,I\}=0. \tag{2.2}$$

The explicit form of the water wave problem in the canonical variables is (see [12])

$$\begin{cases}\partial_t\eta - G(\eta)\xi = 0,\\ \partial_t\xi + g\eta + \dfrac{1}{2}|\partial_x\xi|^2 - \dfrac{(G(\eta)\xi+\partial_x\eta\cdot\partial_x\xi)^2}{2(1+|\partial_x\eta|^2)} = 0.\end{cases} \tag{2.3}$$

This formulation has been extensively used for the mathematical analysis of the initial-value problem as well as in direct numerical simulations and long-wave asymptotics.

The DNO that appears in the water wave equations (2.3) is analytic in η [6] and admits a convergent Taylor series

$$G(\eta) = \sum_{m=0}^{\infty} G^{(m)}(\eta) \tag{2.4}$$

about $\eta = 0$. For each m, the term $G^{(m)}(\eta)$ is homogeneous of degree m in η and can be calculated explicitly via recursive relations. Setting $D = -\mathrm{i}\,\partial_x$, the first three terms are

$$\begin{cases} G^{(0)}(\eta) = |D|, \\ G^{(1)}(\eta) = D \cdot \eta D - G^{(0)} \eta G^{(0)}, \\ G^{(2)}(\eta) = -\dfrac{1}{2}\left(|D|^2 \eta^2 G^{(0)} + G^{(0)} \eta^2 |D|^2 - 2G^{(0)} \eta G^{(0)} \eta G^{(0)}\right). \end{cases}$$

In turn, the Hamiltonian has an expansion in the form

$$H(\eta, \xi) = H^{(2)} + H^{(3)} + H^{(4)} + \dots, \tag{2.5}$$

with

$$\begin{aligned} H^{(2)}(\eta, \xi) &= \frac{1}{2} \int \left(\xi G^{(0)} \xi + g\eta^2\right) \mathrm{d}x, \\ H^{(3)}(\eta, \xi) &= \frac{1}{2} \int \xi G^{(1)} \xi \, \mathrm{d}x = \frac{1}{2} \int \xi \big(D \cdot \eta D - G\eta G^{(0)}\big) \xi \, \mathrm{d}x, \\ H^{(4)}(\eta, \xi) &= \frac{1}{2} \int \xi G^{(2)} \xi \, \mathrm{d}x \\ &= -\frac{1}{4} \int \xi \big(|D|^2 \eta^2 G^{(0)} + G^{(0)} \eta^2 |D|^2 - 2G^{(0)} \eta G^{(0)} \eta G^{(0)}\big) \xi \, \mathrm{d}x, \end{aligned} \tag{2.6}$$

where $H^{(2)}, H^{(3)}, H^{(4)}$ are the quadratic, cubic and quartic parts respectively.

Denote the Fourier transform of the real-valued pair $(\eta(x), \xi(x))$ by

$$(\eta_k, \xi_k) = \frac{1}{(2\pi)^{d/2}} \int e^{-\mathrm{i}k \cdot x} (\eta(x), \xi(x)) \, \mathrm{d}x,$$

where we have dropped the usual "hat" notation and the t-dependence for simplicity. In Fourier variables, the water wave system takes the Hamiltonian form

$$\partial_t \begin{pmatrix} \eta_{-k} \\ \xi_{-k} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_{\eta_k} H \\ \partial_{\xi_k} H \end{pmatrix}. \tag{2.7}$$

3. Calculus of transformations

We recall some basic facts in Hamiltonian theory. Consider a phase space $\mathcal{M}$ with a symplectic structure given by the operator J. Let $H : \mathcal{M} \to \mathbb{R}$ be a

Hamiltonian. A transformation

$$\begin{aligned} \tau : \mathcal{M} &\longrightarrow \mathcal{M} \\ v &\longmapsto w \end{aligned}$$

gives rise to a Hamiltonian defined on $\mathcal{M}$, namely

$$\widetilde{H}(w) = \widetilde{H}(\tau(v)) = H(v).$$

By the chain rule, the Hamiltonian vector field $X_H = J\,\nabla_v H$ on $\mathcal{M}$ is transformed as follows:

$$\partial_t w = (\partial_v \tau)\partial_t v = (\partial_v \tau) J\,\nabla_v H,$$

with

$$\nabla_v H = (\partial_v \tau)^* \nabla_w \widetilde{H},$$

the star denoting the adjoint with respect to the L^2-scalar product. This leads to the evolution equation for w,

$$\partial_t w = (\partial_v \tau) J (\partial_v \tau)^* \nabla_w \widetilde{H} := \widetilde{J}\,\nabla_w \widetilde{H}.$$

The transformation τ induces a symplectic structure on $\mathcal{M}$ given by $\widetilde{J}$ and the transformed vector field $\widetilde{J}\,\nabla_w \widetilde{H}$ is Hamiltonian in the phase space $\mathcal{M}$. When $(\partial_v \tau) J (\partial_v \tau)^* = J$, we say that τ is a canonical transformation. We discuss below a few transformations that play an important role in the derivation of various asymptotic models for the water wave problem, and in particular for the derivation of the Dysthe equation.

In the following, $v = (\eta, \xi)^\top$ and $\mathcal{M} = L^2(\mathbb{R} \times \mathbb{R})$. In each case, we show how the transformation affects the structure map and the corresponding Hamiltonian.

3.1. **Examples of transformations.**

3.1.1. *Amplitude scaling.* Consider the transformation $\tau : v \longmapsto w$ such that

$$w = \begin{pmatrix} \widetilde{\eta} \\ \widetilde{\xi} \end{pmatrix} = \begin{pmatrix} \alpha\eta \\ \beta\xi \end{pmatrix}$$

for α, $\beta \in \mathbb{R}^+$. The Jacobian of this transformation is given by $\partial_v \tau = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, and accordingly the transformed symplectic form is $\widetilde{J} = \alpha\beta J$.

3.1.2. *Hilbert transform.* Let $\tau : v \longmapsto w$ be such that

$$w = \begin{pmatrix} \widetilde{\eta} \\ \widetilde{\xi} \end{pmatrix} = \begin{pmatrix} \mathcal{H}\,\eta \\ \mathcal{H}\,\xi \end{pmatrix},$$

where $\mathcal{H} = -\mathrm{i}\,\mathrm{sgn}(D)$ is the Hilbert transform. Since $\mathcal{H}$ is unitary, the transformation $(\eta, \xi) \longmapsto (\widetilde{\eta}, \widetilde{\xi})$ is canonical.

3.1.3. *Moving reference frame.* It is common to work in coordinate systems that move with a characteristic speed of solutions, namely

$$w(x,t) = v(x-ct,t).$$

However, the time variable t plays a special role, and at first consideration, this transformation, which mixes space and time variables, is not accommodated in the present picture. An alternative is to use that the momentum defined in (2.1) is a conserved quantity and Poisson-commutes with the Hamiltonian as indicated in (2.2). Accordingly, their respective flows also commute:

$$\chi_t^H \circ \chi_s^I(v) = \chi_s^I \circ \chi_t^H(v).$$

Because the vector field associated with the impulse is given by $\partial_s v = J\,\nabla_v I$, or more explicitly,

$$\partial_s \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \xi \\ -\partial_x \eta \end{pmatrix} = \begin{pmatrix} -\partial_x \eta \\ -\partial_x \xi \end{pmatrix},$$

the corresponding flow is simply constant unit-speed translation

$$\chi_s^I(v)(x) = v(x-s),$$

which implies that the flow along the diagonal is

$$\chi_t^H \circ \chi_{-ct}^I(v) = \chi_t^{H-cI}(v).$$

It can be inferred that the Hamiltonian flow of $H(v) - cI(v)$ is the Hamiltonian flow of $H(v)$ observed in a coordinate frame translating with speed c.

3.2. **Complex symplectic coordinates.**

The linear dispersion relation for deep-water gravity waves is

$$\omega_k^2 = g|k|.$$

Introducing the coefficients $a_k := \sqrt[4]{g/|k|}$, we define the complex symplectic coordinates

$$z_k := \frac{1}{\sqrt{2}}(a_k \eta_k + \mathrm{i}\, a_k^{-1} \xi_k). \tag{3.1}$$

Since the functions $\eta(x)$ and $\xi(x)$ are real-valued in the physical space, we also have

$$\bar{z}_{-k} = \frac{1}{\sqrt{2}}(a_k \eta_k - \mathrm{i}\, a_k^{-1} \xi_k).$$

Equivalently, η_k and ξ_k are expressed in terms of z_k as

$$\eta_k = \frac{1}{\sqrt{2}} a_k^{-1}(z_k + \bar{z}_{-k}), \qquad \xi_k = \frac{1}{\sqrt{2}\,\mathrm{i}} a_k^{-1}(z_k - \bar{z}_{-k}). \tag{3.2}$$

In the z_k variables, the system (2.7) now reads

$$\partial_t \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \partial_{z_k} H \\ \partial_{\bar{z}_{-k}} H \end{pmatrix}.$$

The transformation $(\eta_k, \xi_k) \longmapsto (z_k, -\mathrm{i}\, \bar{z}_{-k})$ is canonical. In these variables, the quadratic term $H^{(2)}$ simplifies to

$$H^{(2)} = \int \omega_k |z_k|^2 \, \mathrm{d}k,$$

while the cubic term $H^{(3)}$ becomes

$$H^{(3)} = \frac{1}{8\sqrt{\pi}} \int_{k_1+k_2+k_3=0} (k_1 \cdot k_3 + |k_1||k_3|) \frac{a_1 a_3}{a_2} (z_1 - \bar{z}_{-1})(z_2 + \bar{z}_{-2})(z_3 - \bar{z}_{-3}), \quad (3.3)$$

and the quartic term $H^{(4)}$

$$H^{(4)} = \frac{1}{16\pi} \int_{k_1+k_2+k_3+k_4=0} |k_1||k_4|(|k_1| + |k_4| - 2|k_3 + k_4|) \frac{a_1 a_4}{a_2 a_3} \times (z_1 - \bar{z}_{-1})(z_2 + \bar{z}_{-2})(z_3 + \bar{z}_{-3})(z_4 - \bar{z}_{-4}),$$

where, for simplicity, we use the notation $z_j = z_{k_j}$, $z_{-j} = z_{-k_j}$ and $a_j = a_{k_j}$.

3.3. **Birkhoff normal form.** Birkhoff normal form results from a canonical change of variables up to a given order m, so that the Taylor expansion of the transformed Hamiltonian up to order m contains only resonant terms. Each of the transformations is constructed as the flow of an appropriate auxiliary Hamiltonian system. We perform the construction at third order, which is the one relevant for the present study, where we eliminate all cubic terms leading to a reduced Hamiltonian at fourth order. This order is sufficient for the derivation of the NLS equation and the Dysthe equation.

We seek a transformation

$$\tau : w = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \longmapsto w',$$

defined in a neighborhood of the origin, such that the transformed Hamiltonian satisfies

$$H'(w') = H(\tau^{-1}(w')), \qquad \partial_t w' = J \, \nabla H'(w'),$$

and reduces to

$$H'(w') = H^{(2)}(w') + Z^{(3)} + \cdots + Z^{(m)} + R^{(m+1)},$$

where the terms $Z^{(3)}, \dots, Z^{(m)}$ consist only of resonant terms, and $R^{(m+1)}$ is the remainder term. We construct the transformation τ by the Lie transform method as a Hamiltonian flow ψ from "time" $s = -1$ to "time" $s = 0$ governed by

$$\partial_s \psi = J \, \nabla K(\psi), \quad \psi(w')|_{s=0} = w', \quad \psi(w')|_{s=-1} = w,$$

and associated to an auxiliary Hamiltonian K. Such a transformation is canonical and preserves the Hamiltonian structure of the system. The Hamiltonian H' satisfies $H'(w') = H(\psi(w'))|_{s=-1}$, and its Taylor expansion around $s = 0$ is

$$H'(w') = H(\psi(w'))|_{s=0} - \frac{dH}{ds}(\psi(w'))|_{s=0} + \frac{1}{2} \frac{d^2 H}{ds^2}(\psi(w'))|_{s=0} - \cdots .$$

Abusing notations, we further use $w = (\eta, \xi)^\top$ to denote the new variable w'. Terms in this expansion can be expressed using Poisson brackets as

$$\begin{aligned} H(\psi(w))|_{s=0} &= H(w), \\ \frac{dH}{ds}(\psi(w))|_{s=0} &= \int (\partial_\eta H \partial_s \eta + \partial_\xi H \partial_s \xi)\, \mathrm{d}x, \\ &= \int (\partial_\eta H \partial_\xi K - \partial_\xi H \partial_\eta K)\, \mathrm{d}x = \{K, H\}(w), \end{aligned}$$

and similarly for the remaining terms. The Taylor expansion of H' around $s = 0$ now has the form

$$H'(w) = H(w) - \{K, H\}(w) + \frac{1}{2}\{K, \{K, H\}\}(w) - \cdots .$$

Substituting into the expansion (2.5) of H, we obtain

$$\begin{aligned} H'(w) = {} & H^{(2)}(w) + H^{(3)}(w) + H^{(4)}(w) \ldots \\ & - \{K, H^{(2)}\}(w) - \{K, H^{(3)}\}(w) - \{K, H^{(4)}\}(w) \ldots \\ & + \frac{1}{2}\{K, \{K, H^{(2)}\}\}(w) + \frac{1}{2}\{K, \{K, H^{(3)}\}\}(w) + \cdots . \end{aligned}$$

If K is homogeneous of degree m and $H^{(n)}$ is homogeneous of degree n, then $\{K, H^{(n)}\}$ is of degree $m+n-2$. If we construct an auxiliary Hamiltonian $K = K^{(3)}$ that is homogeneous of degree 3 and satisfies the relation

$$H^{(3)} - \{K^{(3)}, H^{(2)}\} = 0, \tag{3.4}$$

we will have eliminated all cubic terms from the transformed Hamiltonian H'. This process can be repeated potentially at each order. In the following, we focus on the two-dimensional case ($d = 2$).

3.4. **Third-order cohomological equation.** The cohomological relation (3.4) for the unknown Hamiltonian $K^{(3)}$ can be solved in complex symplectic coordinates that diagonalize the linear operation of taking Poisson brackets with $H^{(2)}$. Indeed, taking for example $\mathcal{I} = z_1 z_2 \bar{z}_{-3}$ (a fixed monomial), we have

$$\{\mathcal{I}, H^{(2)}\} = \mathrm{i}\,(\omega_1 + \omega_2 - \omega_3) z_1 z_2 \bar{z}_{-3}. \tag{3.5}$$

Writing $H^{(3)}$ given in (3.3) as a linear combination of third-order monomials in z_k and $\bar{z}_{-k}$, we look for $K^{(3)}$ as a linear combination of the same monomials and identify the coefficients using (3.5). This leads to the following proposition:

Proposition 3.1. *The cohomological equation* (3.4) *has a unique solution, given in complex symplectic coordinates by*

$$\begin{aligned} K^{(3)} = \frac{1}{8\mathrm{i}\sqrt{\pi}} \int_{k_1+k_2+k_3=0} & (k_1 k_3 + |k_1||k_3|) \frac{a_1 a_3}{a_2} \Big[\frac{z_1 z_2 z_3 - \bar{z}_{-1} \bar{z}_{-2} \bar{z}_{-3}}{\omega_1 + \omega_2 + \omega_3} \\ & - 2 \frac{z_1 z_2 \bar{z}_{-3} - \bar{z}_{-1} \bar{z}_{-2} z_3}{\omega_1 + \omega_2 - \omega_3} + \frac{z_1 \bar{z}_{-2} z_3 - \bar{z}_{-1} z_2 \bar{z}_{-3}}{\omega_1 - \omega_2 + \omega_3} \Big]. \end{aligned} \tag{3.6}$$

In the variables (η_k, ξ_k), $K^{(3)}$ takes the form

$$K^{(3)} = \frac{1}{\sqrt{2\pi}} \int_{k_1+k_2+k_3=0} \frac{k_1k_3 + |k_1||k_3|}{d(\omega_1,\omega_2,\omega_3)} \Big[a_1^2 \omega_1 (\omega_1^2 - \omega_2^2 - \omega_3^2) \eta_1\eta_2\xi_3 + \frac{a_1^2 a_3^2}{a_2^2} \omega_1\omega_2\omega_3 \eta_1\xi_2\eta_3 + \frac{1}{2a_2^2} \omega_2 (\omega_1^2 - \omega_2^2 + \omega_3^2) \xi_1\xi_2\xi_3 \Big],$$

where the denominator $d(\omega_1,\omega_2,\omega_3)$ is given by

$$d(\omega_1,\omega_2,\omega_3) = (\omega_1+\omega_2+\omega_3)(\omega_1+\omega_2-\omega_3)(\omega_1-\omega_2+\omega_3)(\omega_1-\omega_2-\omega_3).$$

An important point is to check that the integrals are convergent, meaning that the denominators remain away from zero for the triads $(k_1, k_2, k_3) \in \mathbb{R}^3$, with $k_1 + k_2 + k_3 = 0$. In the periodic setting, where wavenumbers take discrete values, it is known that there are no resonant triads, i.e., there are no triplets (k_1, k_2, k_3), with $k_j \neq 0$ positive or negative integers, such that $k_1 + k_2 + k_3 = 0$ and $\sqrt{|k_1|} \pm \sqrt{|k_2|} \pm \sqrt{|k_3|} = 0$ for any choice of sign. This is due to the increasing and concave character of the dispersion relation $\omega_k = \sqrt{g|k|}$.

When the problem is considered on the whole real line, the denominators of $K^{(3)}$ could potentially be close to zero for some triplets $(k_1, k_2, k_3) \in \mathbb{R}^3$, with $k_1 + k_2 + k_3 = 0$. However, the term $k_1k_3 + |k_1||k_3|$ in the numerator simplifies to

$$k_1k_3 + |k_1||k_3| = |k_1||k_3|(1 + \operatorname{sgn}(k_1)\operatorname{sgn}(k_3)).$$

This implies that the domain of integration is reduced to triplets $(k_1, k_2, k_3) \in \mathbb{R}^3$ with $k_1 + k_2 + k_3 = 0$ and k_1 and k_3 having the same sign. In this region of the (k_1, k_3)-plane, the denominator $d(\omega_1,\omega_2,\omega_3)$ becomes (see [13])

$$d(\omega_1,\omega_2,\omega_3) = -4g^2 k_1 k_3.$$

Proposition 3.2. *The expression for $K^{(3)}$ simplifies to*

$$K^{(3)} = -\frac{1}{4\sqrt{2\pi}} \int_{k_1+k_2+k_3=0} (1 + \operatorname{sgn}(k_1)\operatorname{sgn}(k_3))(-2|k_3|\eta_1\eta_2\xi_3 + |k_2|\eta_1\xi_2\eta_3).$$

We see in particular that there is no problem of small denominators in the definition of $K^{(3)}$. A further simplification occurs when one uses the following relation, which can be checked by considering the different regions of the integration domain.

Lemma 3.3. *For any $(k_1, k_2, k_3) \in \mathbb{R}^3$ such that $k_1 + k_2 + k_3 = 0$, we have*

$$\operatorname{sgn}(k_1)\operatorname{sgn}(k_2) + \operatorname{sgn}(k_2)\operatorname{sgn}(k_3) + \operatorname{sgn}(k_3)\operatorname{sgn}(k_1) = -1.$$

Finally, we arrive to a simple expression for $K^{(3)}$ written in Fourier or physical space.

Proposition 3.4. *The expression for $K^{(3)}$ is given by*

$$K^{(3)} = -\frac{1}{2\sqrt{2\pi}} \int_{k_1+k_2+k_3=0} \operatorname{sgn}(k_1)\operatorname{sgn}(k_2)\, \eta_1\eta_2 |k_3| \xi_3. \tag{3.7}$$

Equivalently, in the physical space,

$$K^{(3)} = \frac{1}{2}\int (-\mathrm{i}\,\mathrm{sgn}(D)\eta)^2 |D|\xi \,\mathrm{d}x. \tag{3.8}$$

Remark 3.5. A formula for $K^{(3)}$ similar to (3.6) holds in the three-dimensional case with the product $k_1 k_3$ replaced by scalar product $k_1 \cdot k_3$, and is given in [24]. Because the wavenumbers k_j are in $\mathbb{R}^2$, the expression does not simplify. In the integrals, any of the k_j's could be arbitrarily close to 0. However one checks that, although a denominator could become very small, it is compensated by a small numerator and all integrals are convergent.

3.5. **Third-order normal form.** The third-order normal form defining the new coordinates is obtained as the solution map at $s = 0$ of the Hamiltonian flow

$$\partial_s \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_\eta K^{(3)} \\ \partial_\xi K^{(3)} \end{pmatrix},$$

with initial condition at $s = -1$ being the original variables (η, ξ). Equivalently, in Fourier coordinates,

$$\partial_s \eta_{-k} = \partial_{\xi_k} K^{(3)}, \qquad \partial_s \xi_{-k} = -\partial_{\eta_k} K^{(3)}.$$

The expressions (3.7) and (3.8) for $K^{(3)}$ invite us to make another change of variables, namely $(\eta, \xi) \longmapsto (\widetilde{\eta}, \widetilde{\xi})$, where

$$\widetilde{\eta} = \mathcal{H}\,\eta = -\mathrm{i}\,\mathrm{sgn}(D)\eta, \qquad \widetilde{\xi} = \mathcal{H}\,\eta = -\mathrm{i}\,\mathrm{sgn}(D)\xi. \tag{3.9}$$

As discussed in Section 3.1.2, this Hilbert transformation is canonical, and $K^{(3)}$ reads

$$K^{(3)} = \frac{1}{2}\int (\widetilde{\eta})^2 \partial_x \widetilde{\xi} \,\mathrm{d}x$$

in terms of variables $(\widetilde{\eta}, \widetilde{\xi})$. The associated Hamiltonian flow is simply

$$\partial_s \widetilde{\eta} = \partial_{\widetilde{\xi}} K^{(3)} = -\frac{1}{2}\partial_x (\widetilde{\eta})^2,$$
$$\partial_s \widetilde{\xi} = -\partial_{\widetilde{\eta}} K^{(3)} = -\widetilde{\eta}\partial_x \widetilde{\xi}.$$

Theorem 3.6. *The Hamiltonian system that defines the third-order Birkhoff normal form transformation has the form of two coupled PDEs*

$$\partial_s \widetilde{\eta} + \widetilde{\eta}\partial_x \widetilde{\eta} = 0, \tag{3.10a}$$
$$\partial_s \widetilde{\xi} + \widetilde{\eta}\partial_x \widetilde{\xi} = 0, \tag{3.10b}$$

where $(\widetilde{\eta}, \widetilde{\xi})$ are the Hilbert transforms of (η, ξ). The equation for $\widetilde{\eta}$ is an inviscid Burgers equation. The second equation for $\widetilde{\xi}$ is its linearization along Burgers' flow.

4. Transformed Hamiltonian

After applying the third-order normal form transformation, the new Hamiltonian H' becomes (with the prime dropped)

$$\begin{aligned} H &= H^{(2)} + H^{(4)} - \{K^{(3)}, H^{(3)}\} + \frac{1}{2}\{K^{(3)}, \{K^{(3)}, H^{(2)}\}\} + R^{(5)} \\ &= H^{(2)} + H_+^{(4)} + R^{(5)}, \end{aligned}$$

where $R^{(5)}$ denotes all terms of order 5 and higher and $H_+^{(4)}$ is the new fourth-order term

$$H_+^{(4)} = H^{(4)} - \frac{1}{2}\{K^{(3)}, H^{(3)}\}.$$

Lemma 4.1 ([13]). *The correction term $\frac{1}{2}\{K^{(3)}, H^{(3)}\}$ takes the special form*

$$\frac{1}{2}\{K^{(3)}, H^{(3)}\} = H^{(4)}(\widetilde{\eta}, \xi).$$

It has the same form as the original $H^{(4)}$ given in (2.6) *with η replaced by $\widetilde{\eta} = -\mathrm{i}\,\mathrm{sgn}(D)\eta$.*

Proof. We first calculate the gradients of $K^{(3)}$ and $H^{(3)}$. From (3.8), we have

$$\begin{aligned} \partial_\eta K^{(3)} &= -(-\mathrm{i}\,\mathrm{sgn}(D))[(-\mathrm{i}\,\mathrm{sgn}(D)\eta)|D|\xi] = \mathrm{i}\,\mathrm{sgn}(D)(\widetilde{\eta}|D|\xi), \\ \partial_\xi K^{(3)} &= \frac{1}{2}|D|(-\mathrm{i}\,\mathrm{sgn}(D)\eta)^2 = \frac{1}{2}|D|(\widetilde{\eta})^2. \end{aligned}$$

From (2.6) rewritten as

$$H^{(3)} = \frac{1}{2}\int \eta\Big((\partial_x\xi)^2 - (|D|\xi)^2\Big)\mathrm{d}x,$$

we have

$$\begin{aligned} \partial_\eta H^{(3)} &= \frac{1}{2}\Big((\partial_x\xi)^2 - (|D|\xi)^2\Big) = \mathrm{i}\,\mathrm{sgn}(D)((\partial_x\xi)|D|\xi), \\ \partial_\xi H^{(3)} &= -\partial_x(\eta\partial_x\xi) - |D|(\eta|D|\xi). \end{aligned}$$

Computing the Poisson bracket $\{K^{(3)}, H^{(3)}\}$ we have, after some calculations,

$$\{K^{(3)}, H^{(3)}\} = \int (\widetilde{\eta}|D|\xi)\Big(|D|(\eta\partial_x\xi) - \partial_x(\eta|D|\xi) - \partial_x\widetilde{\eta}\,\partial_x\xi\Big)\mathrm{d}x. \tag{4.1}$$

This expression further simplifies using the following observation. If $F(x) = f(x) + \mathrm{i}\,g(x)$ is the boundary value of a holomorphic function $F(z)$ in the upper half-plane, i.e., is in a Hardy space, then g is the Hilbert transform of f, equivalently $g = -\mathrm{i}\,\mathrm{sgn}(D)f$. When $(f_1 + \mathrm{i}\,g_1)$ and $(f_2 + \mathrm{i}\,g_2)$ are functions in a Hardy space, their product $fg = (f_1f_2 - g_1g_2) + \mathrm{i}\,(f_1g_2 + f_2g_1)$ is also in a Hardy space. Therefore, setting $\widetilde{f}_j = -\mathrm{i}\,\mathrm{sgn}(D)f_j$ $(j = 1, 2)$ we have

$$f_1f_2 - \widetilde{f}_1\widetilde{f}_2 = \mathrm{i}\,\mathrm{sgn}(D)(f_1\widetilde{f}_2 + f_2\widetilde{f}_1). \tag{4.2}$$

Identity (4.2) with $f_1 = \eta$ and $f_2 = \partial_x\xi$ implies

$$\eta\partial_x\xi - \widetilde{\eta}\partial_x\widetilde{\xi} = \mathrm{i}\,\mathrm{sgn}(D)(\eta\partial_x\widetilde{\xi} + \widetilde{\eta}\partial_x\xi).$$

Applying $|D|$ to both sides and using that $\partial_x \widetilde{\xi} = |D|\xi$, we have

$$|D|(\eta\partial_x\xi - \widetilde{\eta}|D|\xi) = \partial_x(\eta|D|\xi + \widetilde{\eta}\partial_x\xi).$$

Equivalently,

$$|D|(\eta\partial_x\xi) - \partial_x(\eta|D|\xi) = |D|(\widetilde{\eta}|D|\xi) + \partial_x(\widetilde{\eta}\partial_x\xi).$$

Substituting in (4.1), we get

$$\{K^{(3)}, H^{(3)}\} = \int \Big((\widetilde{\eta}|D|\xi)|D|(\widetilde{\eta}|D|\xi) + (\widetilde{\eta})^2\partial_x^2\xi|D|\xi \Big) \mathrm{d}x,$$

which can be further rewritten as

$$\{K^{(3)}, H^{(3)}\} = -\frac{1}{2}\int \Big(-2(\widetilde{\eta}|D|\xi)|D|(\widetilde{\eta}|D|\xi) + \xi|D|(\widetilde{\eta}^2 D^2\xi) + \xi D^2(\widetilde{\eta}^2|D|\xi)\Big)\, \mathrm{d}x.$$

This formula coincides with the expression $\int \xi G^{(2)}(\widetilde{\eta})\xi\, \mathrm{d}x$. □

As a consequence, we have the following proposition:

Proposition 4.2. *The fourth-order term $H_+^{(4)}$ of H is*

$$H_+^{(4)} = H^{(4)}(\eta, \xi) - H^{(4)}(\widetilde{\eta}, \xi). \tag{4.3}$$

Remark 4.3. The simplicity of the expression for $K^{(3)}$ allows us to perform the above calculation in the physical space with variables (η, ξ). This is not the case in three dimensions as shown in [24], where calculations are done in terms of complex symplectic coordinates in the Fourier space.

Proposition 4.4. *The resulting Hamiltonian system is, after the elimination of all cubic terms in the Hamiltonian,*

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_\eta H \\ \partial_\xi H \end{pmatrix}, \tag{4.4}$$

with H being the truncated Hamiltonian $H = H^{(2)} + H_+^{(4)}$ and $H_+^{(4)}$ given by (4.3)*.*

5. Hamiltonian Dysthe equation

5.1. **Modulational Ansatz.** We first return to the complex symplectic coordinates $(z, \bar{z})$ defined in (3.1). As a result, the system (4.4) becomes

$$\partial_t \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = J_1 \begin{pmatrix} \partial_z H \\ \partial_{\bar{z}} H \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \partial_z H \\ \partial_{\bar{z}} H \end{pmatrix}.$$

We now introduce the modulational Ansatz that captures the slow modulation of small-amplitude almost monochromatic waves with carrier wavenumber $k_0 > 0$ by assuming that

$$z = \varepsilon u(X, t) e^{\mathrm{i}k_0 x}, \qquad X = \varepsilon x.$$

The small dimensionless parameter $\varepsilon \sim k_0 A_0 \ll 1$ is a measure of the wave steepness (A_0 being a characteristic amplitude of the free surface). Equivalently, it is a measure of the wave spectrum's narrowness around $k = k_0$. In the framework of transformation theory,

$$\begin{pmatrix} u \\ \bar{u} \end{pmatrix} = P_2 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \varepsilon^{-1} \begin{pmatrix} e^{-\mathrm{i}k_0 x} & 0 \\ 0 & e^{\mathrm{i}k_0 x} \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}. \tag{5.1}$$

The corresponding equations of motion are

$$\partial_t \begin{pmatrix} u \\ \bar{u} \end{pmatrix} = J_2 \begin{pmatrix} \partial_u H \\ \partial_{\bar{u}} H \end{pmatrix} = \varepsilon^{-1} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \partial_u H \\ \partial_{\bar{u}} H \end{pmatrix}, \tag{5.2}$$

where $J_2 = \varepsilon P_2 J_1 P_2^*$ [9].

5.2. **Hamiltonian in transformed coordinates.** We now implement the modulational Ansatz (5.1) in the Hamiltonian. Two important elements of analysis come into play. First, due to the multiscale nature of this problem (fast oscillations in x and slow modulation in X), we need to examine the action of Fourier multiplier operators on multiscale functions.

Proposition 5.1. *Let $m(D)$ be a Fourier multiplier. For sufficiently smooth functions $f(X)$, we have*

$$\begin{aligned} m(D)(e^{\mathrm{i}k_0 x} f(X)) = e^{\mathrm{i}k_0 x} \Big(& m(k_0) + \varepsilon \partial_k m(k_0) D_X f(X) + \frac{\varepsilon^2}{2} \partial_k^2 m(k_0) D_X^2 f(X) \\ & + \cdots + \frac{\varepsilon^N}{N!} \partial_k^N m(k_0) D_X^N f(X) \Big) + R^{(N+1)}, \end{aligned}$$

where $R^{(N+1)}$ is a remainder of order $O(\varepsilon^{N+1})$.

Precise estimates of the remainder, as well as generalization to pseudo-differential operators $m = m(x, \varepsilon x, D)$, can be found in [14]. We apply the proposition to $\omega(D) = (g|D|)^{1/2}$ acting on $e^{\mathrm{i}k_0 x} u(X)$ and find

$$\begin{aligned} \omega(D) \left(e^{\mathrm{i}k_0 x} u(X) \right) &= e^{\mathrm{i}k_0 x} \omega(k_0 + \varepsilon D_X) u, \\ &= e^{\mathrm{i}k_0 x} \Big(\omega_0 + \varepsilon \omega_0' D_X + \frac{\varepsilon^2}{2} \omega_0'' D_X^2 + \frac{\varepsilon^3}{6} \omega_0''' D_X^3 \Big) u + O(\varepsilon^4), \end{aligned}$$

where we have used the notation $\omega_0 = \omega(k_0)$ and similarly for its derivatives. As a consequence, the quadratic part $H^{(2)}$ of the Hamiltonian is expanded as

$$\begin{aligned} H^{(2)} &= \varepsilon \int \bar{u}\, \omega(k_0 + \varepsilon D_X) u \, \mathrm{d}X, \\ &= \frac{\varepsilon}{2} \int \bar{u} \Big(\omega_0 + \varepsilon \omega_0' D_X + \frac{\varepsilon^2}{2} \omega_0'' D_X^2 + \frac{\varepsilon^3}{6} \omega_0''' D_X^3 \Big) u \, \mathrm{d}X + \text{c.c.}, \end{aligned}$$

where 'c.c.' stands for the complex conjugate of all the preceding terms.

Secondly, terms in $H_+^{(4)}$ that appear with fast oscillations essentially homogenize to zero and thus do not contribute to the effective Hamiltonian. This is a consequence of the following scale separation lemma.

Proposition 5.2. *Let $g(x)$ be a function on $\mathbb{R}$, periodic over a fundamental domain $[0, 2\pi)$, and denote by $\overline{g}$ its average over the period. For any function $f(X)$ in the Schwartz space, we have*

$$\int g(X/\varepsilon) f(X)\,\mathrm{d}X = \overline{g} \int f(X)\,\mathrm{d}X + O(\varepsilon^N)$$

for any N.

The proof of this proposition can be found in [8]. This homogenization naturally selects in $H_+^{(4)}$ 4-wave resonances among all the possible quartic interactions as the terms presenting fast oscillations exactly cancel out. Consequently, the quartic term $H_+^{(4)}$ reduces to the sum of the two quantities below:

$$\begin{aligned} H^{(4)}(\eta,\xi) &= \frac{\varepsilon^3}{4} \int \Big[k_0^3 |u|^4 + \frac{3}{2}\varepsilon k_0^2 |u|^2 \left(\bar{u} D_X u + u \overline{D_X u} \right) \Big]\,\mathrm{d}X, \\ H^{(4)}(-\mathrm{i}\,\mathrm{sgn}(D)\eta,\xi) &= -\frac{\varepsilon^3}{4} \int \Big[k_0^3 |u|^4 + \frac{3}{2}\varepsilon k_0^2 |u|^2 \left(\bar{u} D_X u + u \overline{D_X u} \right) \\ &\qquad - 2\varepsilon k_0^2 |u|^2 |D_X| |u|^2 \Big]\,\mathrm{d}X. \end{aligned}$$

At order $O(\varepsilon^4)$, the reduced Hamiltonian reads

$$\begin{aligned} H = \frac{\varepsilon}{2} \int & \bar{u}\Big(\omega_0 + \varepsilon \omega_0' D_X + \frac{\varepsilon^2}{2} \omega_0'' D_X^2 + \frac{\varepsilon^3}{6} \omega_0''' D_X^3 \Big) u + \text{c.c.} \\ & + \varepsilon^2 k_0^3 |u|^4 + \frac{3}{2}\varepsilon^3 k_0^2 |u|^2 \left(\bar{u} D_X u + u \overline{D_X u} \right) - \varepsilon^3 k_0^2 |u|^2 |D_X| |u|^2 \,\mathrm{d}X + O(\varepsilon^5), \end{aligned}$$

or equivalently,

$$\begin{aligned} H = \varepsilon \int & \omega_0 |u|^2 + \varepsilon \omega_0' \operatorname{Im}(\bar{u}\partial_X u) + \frac{\varepsilon^2}{2}\omega_0'' |\partial_X u|^2 + \frac{\varepsilon^2}{2} k_0^3 |u|^4 + \frac{\varepsilon^3}{6}\omega_0''' \operatorname{Im}\big[(\overline{\partial_X u})(\partial_X^2 u) \big] \\ & + \frac{3}{2}\varepsilon^3 k_0^2 |u|^2 \operatorname{Im}(\bar{u}\partial_X u) - \frac{\varepsilon^3}{2} k_0^2 |u|^2 |D_X| |u|^2 \,\mathrm{d}X + O(\varepsilon^5). \end{aligned} \tag{5.3}$$

5.3. **Derivation of the Dysthe equation.** The evolution equation (5.2) for u becomes

$$\begin{aligned} \partial_t u &= -\mathrm{i}\varepsilon^{-1} \partial_{\bar{u}} H \\ &= -\mathrm{i}\omega_0 u - \varepsilon \omega_0' \partial_X u + \mathrm{i}\frac{\varepsilon^2}{2}\omega_0'' \partial_X^2 u - \mathrm{i}\varepsilon^2 k_0^3 |u|^2 u \\ &\quad + \frac{\varepsilon^3}{6}\omega_0''' \partial_X^3 u - 3\varepsilon^3 k_0^2 |u|^2 \partial_X u + \mathrm{i}\varepsilon^3 k_0^2 u |D_X| |u|^2, \end{aligned} \tag{5.4}$$

which is a Hamiltonian version of Dysthe's equation for deep-water gravity waves (see [20]), associated with the Hamiltonian (5.3).

The first two terms on the right-hand side of (5.4) can be eliminated by phase invariance and reduction into a moving reference frame. The latter transformation is equivalent, in the framework of canonical transformations, to the substraction from H of a multiple of the momentum

$$I = \varepsilon \int k_0|u|^2 + \varepsilon \operatorname{Im}(\bar{u}\partial_X u)\,\mathrm{d}X$$

(see Section 3.1), while the former is equivalent to the substraction from H of a multiple of the wave action

$$M = \varepsilon \int |u|^2\,\mathrm{d}X, \tag{5.5}$$

which is also conserved due to the phase-invariance property of the Dysthe equation. Finally, the Hamiltionian takes the form

$$\widehat{H} = H - \omega_0' I - (\omega_0 - k_0\omega_0')M,$$

which, after introducing a new long-time scale $\tau = \varepsilon^2 t$, leads to the Hamiltonian Dysthe equation

$$\mathrm{i}\partial_\tau u = \frac{\omega_0}{8k_0^2}\partial_X^2 u + k_0^3|u|^2 u + \mathrm{i}\varepsilon\frac{\omega_0}{16k_0^3}\partial_X^3 u - 3\mathrm{i}\varepsilon k_0^2|u|^2\partial_X u - \varepsilon k_0^2 u|D_X||u|^2.$$

It describes the evolution of the envelope of modulated waves moving in the positive horizontal direction at group velocity $\omega_0' = \omega_0/(2k_0)$. The operator $|D_X|$ is the Fourier multiplier with symbol $|k|$. The associated Hamiltonian is

$$\begin{aligned} H = \int \omega_0|u|^2 &+ \varepsilon\frac{\omega_0}{2k_0}\operatorname{Im}(\bar{u}\partial_X u) - \varepsilon^2\frac{\omega_0}{8k_0^2}|\partial_X u|^2 + \varepsilon^2\frac{k_0^3}{2}|u|^4 \\ &+ \varepsilon^3\frac{\omega_0}{16k_0^3}\operatorname{Im}\big[(\overline{\partial_X u})(\partial_X^2 u)\big] + \varepsilon^3\frac{3k_0^2}{2}|u|^2\operatorname{Im}(\bar{u}\partial_X u) - \varepsilon^3\frac{k_0^2}{2}|u|^2|D_X||u|^2\,\mathrm{d}X. \end{aligned}$$

A remarkable property of the Dysthe equation is the presence of the nonlocal term $u|D_X||u|^2$ that signals the presence of a wave-induced mean flow. In the present derivation, it originates from the third-order normal form that eliminates cubic terms in the Hamiltonian. In previous derivations of the Dysthe equation, it comes from an auxiliary Laplace problem in a fixed domain [37, 38]. Unlike the classical Dysthe equation, the Hamiltonian Dysthe equation does not have a term of the form $u^2\partial_X\bar{u}$. It would be of interest to examine whether the well-posedness results of [24] and [32] obtained for the classical Dysthe equation extend to its Hamiltonian version.

5.4. **Dysthe equation with exact dispersion.** An equivalent, potentially better approximation for numerical purposes of small-amplitude water waves in a modulational regime, consists in keeping the linear dispersion relation $\omega(k_0 + \varepsilon D_X)$ unchanged, while expanding the nonlinear contributions up to order $O(\varepsilon^4)$ as above. The resulting Dysthe equation with exact linear dispersion is

$$\partial_t u = -\mathrm{i}\omega(k_0 + \varepsilon D_X)u - \mathrm{i}\varepsilon^2 k_0^3|u|^2 u - 3\varepsilon^3 k_0^2|u|^2\partial_X u + \mathrm{i}\varepsilon^3 k_0^2 u|D_X||u|^2, \tag{5.6}$$

and the corresponding Hamiltonian is

$$H = \varepsilon \int \bar{u}\,\omega(k_0 + \varepsilon D_X)u + \frac{\varepsilon^2}{2}k_0^3|u|^4 + \frac{3}{2}\varepsilon^3 k_0^2|u|^2\,\mathrm{Im}(\bar{u}\partial_X u) - \frac{\varepsilon^3}{2}k_0^2|u|^2|D_X||u|^2\,\mathrm{d}X. \tag{5.7}$$

In the next section, we explore this question and test this model against direct numerical simulations of the full water wave system.

5.5. **Hamiltonian Dysthe equation for three-dimensional water waves.** For three-dimensional gravity waves on deep water, the Hamiltonian version of Dysthe's equation takes the form (see [24])

$$\begin{aligned} \mathrm{i}\partial_t u &= \omega_0 u - \mathrm{i}\varepsilon\frac{\omega_0}{2k_0}\partial_X u + \varepsilon^2\frac{\omega_0}{8k_0^2}\partial_X^2 u - \varepsilon^2\frac{\omega_0}{4k_0^2}\partial_Y^2 u + \varepsilon^2 k_0^3|u|^2 u \\ &\quad + \mathrm{i}\varepsilon^3\frac{\omega_0}{16k_0^3}\partial_X^3 u - \mathrm{i}\varepsilon^3\frac{3\omega_0}{8k_0^3}\partial_X\partial_Y^2 u - 3\mathrm{i}\varepsilon^3 k_0^2|u|^2\partial_X u + \varepsilon^3 k_0^2 u\,\partial_X^2|D|^{-1}|u|^2. \end{aligned} \tag{5.8}$$

It describes modulated waves moving primarily in the positive X-direction, across the horizontal (X, Y)-plane, at group velocity $\partial_{k_x}\omega(k_0) = \omega_0/(2k_0)$ as shown by the advection term. The nonlinear terms are similar to those in the two-dimensional case but in the nonlocal term $u\,\partial_X^2|D|^{-1}|u|^2$, $|D|$ is the Fourier multiplier with symbol $\sqrt{k_x^2 + k_y^2}$. It is the only place where the dependence on k_y appears in the nonlinear terms. The derivation itself is more involved than the one in two dimensions, as detailed in [24]. This is due to the fact that wavenumbers have two components, and the simple form of the operator $K^{(3)}$ and its Poisson bracket $\{H^{(3)}, K^{(3)}\}$ no longer holds.

Like in two dimensions, a phase shift together with the introduction of a slow time $\tau = \varepsilon^2 t$ in a moving reference frame leads to

$$\begin{aligned} \mathrm{i}\partial_\tau u &= \frac{\omega_0}{8k_0^2}\partial_X^2 u - \frac{\omega_0}{4k_0^2}\partial_Y^2 u + k_0^3|u|^2 u \\ &\quad + \mathrm{i}\varepsilon\frac{\omega_0}{16k_0^3}\partial_X^3 u - \mathrm{i}\varepsilon\frac{3\omega_0}{8k_0^3}\partial_X\partial_Y^2 u - 3\mathrm{i}\varepsilon k_0^2|u|^2\partial_X u + \varepsilon k_0^2 u\,\partial_X^2|D|^{-1}|u|^2. \end{aligned}$$

The associated Hamiltonian is

$$\begin{aligned} H = \int &\omega_0|u|^2 + \varepsilon\frac{\omega_0}{2k_0}\,\mathrm{Im}(\bar{u}\partial_X u) - \varepsilon^2\frac{\omega_0}{8k_0^2}|\partial_X u|^2 + \varepsilon^2\frac{\omega_0}{4k_0^2}|\partial_Y u|^2 + \varepsilon^2\frac{k_0^3}{2}|u|^4 \\ &+ \varepsilon^3\frac{\omega_0}{16k_0^3}\,\mathrm{Im}\big[(\overline{\partial_X u})(\partial_X^2 u)\big] - \varepsilon^3\frac{3\omega_0}{8k_0^3}\,\mathrm{Im}\big[(\overline{\partial_X u})(\partial_Y^2 u)\big] \\ &+ \varepsilon^3\frac{3k_0^2}{2}|u|^2\,\mathrm{Im}(\bar{u}\partial_X u) + \varepsilon^3\frac{k_0^2}{2}|u|^2\partial_X^2|D|^{-1}|u|^2\,\mathrm{d}X\mathrm{d}Y. \end{aligned}$$

5.6. **Reconstruction of the free surface.** In the modulational approximation, an additional step is required in order to reconstruct the free surface from the wave envelope. In the classical theory, this reconstruction is carried out perturbatively in terms of a Stokes expansion, by adding contributions from dominant harmonics of the wave spectrum.

Our Hamiltonian approach performs the surface reconstruction by inverting the transformations associated with our modulational Ansatz and the third-order normal form that eliminates $H^{(3)}$. In two dimensions ($d = 2$), at any instant t, the conversion of η back to its original definition is governed by the inviscid Burgers equation

$$\partial_s \widetilde{\eta} + \widetilde{\eta} \partial_x \widetilde{\eta} = 0 \tag{5.9}$$

for $s \in [-1, 0)$, with "initial" condition

$$\eta(x,t)\big|_{s=0} = \frac{\varepsilon}{\sqrt{2}} a^{-1}(D) \left[u(X,t) e^{\mathrm{i} k_0 x} + \bar{u}(X,t) e^{-\mathrm{i} k_0 x} \right]. \tag{5.10}$$

The choice of this initial condition follows from (3.2) and (5.1). Equation (5.9) may be solved numerically in combination with the envelope equation for u. At any instant t, the final solution of (5.9) at $s = -1$ represents the original physical variable η, with η and $\widetilde{\eta}$ being related through (3.9). Starting from (5.10), Eq. (5.9) will generate higher-order contributions from lower (i.e., mean-flow) and higher harmonics through the nonlinear interactions. The fact that the initial condition (5.10) is sufficiently small and smooth ensures that no shock can form yet during the short interval $s \in [-1, 0)$.

6. Numerical simulations

To illustrate the performance of our Hamiltonian approach, we focus on model (5.6) with exact linear dispersion in the two-dimensional case and perform numerical simulations in the context of modulational (or Benjamin–Feir) instability of Stokes waves. We refer the reader to [10] and [24] for a detailed account of numerical tests on the Hamiltonian Dysthe equation (5.4) and (5.8) (with truncated linear dispersion) in the two- and three-dimensional case, respectively.

6.1. **Stability of Stokes waves.** We first recall the theoretical prediction for this stability problem. Inserting in (5.6) a perturbation of the form

$$u(X,t) = u_0(t) \big[1 + B(X,t) \big]$$

to the uniform wave train solution

$$u_0(t) = B_0 e^{-\mathrm{i}(\omega_0 + \varepsilon^2 k_0^3 B_0^2) t},$$

where

$$B(X,t) = B_1 e^{\Omega t + \mathrm{i} \lambda X} + B_2 e^{\overline{\Omega} t - \mathrm{i} \lambda X},$$

we find that the condition $\mathrm{Re}(\Omega) \neq 0$ for linear instability implies

$$\alpha = -\big[\mathcal{D}(\lambda) + \mathcal{D}(-\lambda) \big]^2 + 4 \varepsilon^2 k_0^2 B_0^2 \big[\mathcal{D}(\lambda) + \mathcal{D}(-\lambda) \big] \left(k_0 - \varepsilon |\lambda| \right) > 0, \tag{6.1}$$

in terms of the exact linear multiplier

$$\mathcal{D}(\lambda) = \omega_0 - \omega(k_0 + \varepsilon \lambda) = \sqrt{g k_0} - \sqrt{g |k_0 + \varepsilon \lambda|}.$$

The slight "Doppler shift" relative to k_0 as induced by the mean flow is clearly noticeable in (6.1).

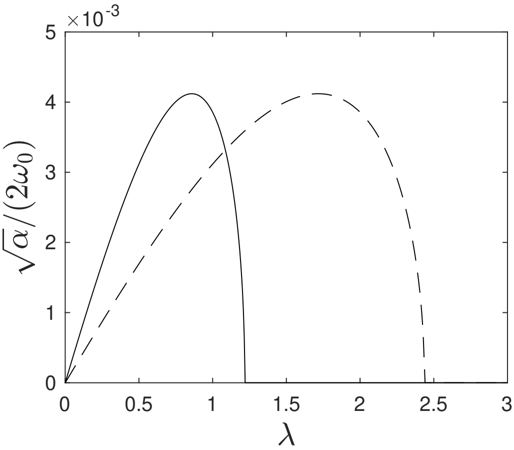


FIGURE 1. Growth rate associated with modulational instability for (5.6). $(A_0, k_0) = (0.02, 5)$ (solid line) and $(0.01, 10)$ (dashed line).

Figure 1 shows the normalized growth rate

$$\frac{|\operatorname{Re}(\Omega)|}{\omega_0} = \frac{\sqrt{\alpha}}{2\omega_0},$$

associated with instability condition (6.1) for two cases $(A_0, k_0) = (0.02, 5)$ and $(0.01, 10)$ corresponding to the same initial steepness $\varepsilon = 0.1$. Hereafter, all the variables are rescaled to absorb ε back into their definition, and all the equations are non-dimensionalized so that $g = 1$. The envelope amplitude in (6.1) is specified as

$$B_0 = A_0 \sqrt[4]{\frac{g}{4k_0}},$$

according to (3.1) and (5.1). As expected, this instability occurs at sideband wavenumbers λ near zero. Maximum instability (i.e., maximum growth rate) is predicted at $\lambda \simeq 1$ for $(A_0, k_0) = (0.02, 5)$ and at $\lambda \simeq 2$ for $(A_0, k_0) = (0.01, 10)$.

6.2. **Numerical methods.** The Hamiltonian Dysthe equation (5.6) with exact linear dispersion is solved numerically using a pseudo-spectral method to discretize in space. The computational domain spans the interval $0 \leq x \leq L$ with periodic boundary conditions and is divided into a regular grid of N collocation points. Use of the FFT leads to efficient and accurate computations, especially when evaluating Fourier multipliers. Time integration is carried out in the Fourier space so that the linear dispersion can be accommodated exactly by the integrating factor technique. The nonlinear terms are integrated in time by using a fourth-order Runge–Kutta scheme with constant step Δt.

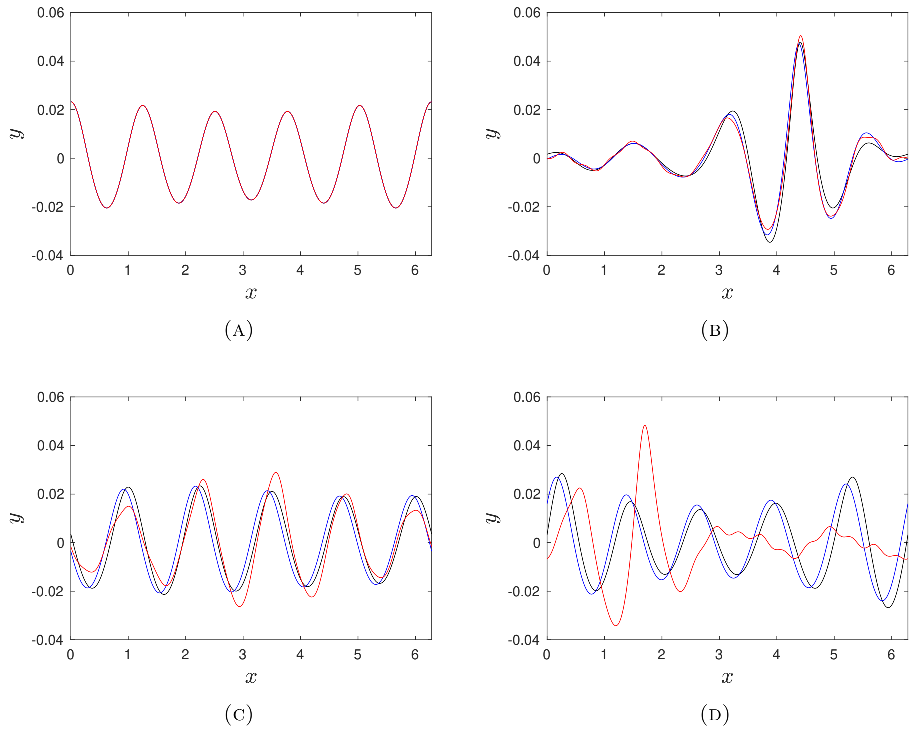


FIGURE 2. Comparison on η at (A) $t = 0$, (B) $t = 370$, (C) $t = 820$, (D) $t = 1000$ for $A_0 = 0.02$, $k_0 = 5$, $\lambda_0 = 1$. Black: full water wave system; blue: Hamiltonian Dysthe with exact dispersion; red: Hamiltonian Dysthe with truncated dispersion.

In the present approach, the third-order normal form transformation not only helps eliminate all non-resonant cubic terms in a manner consistent with the Hamiltonian viewpoint, but in turn also provides a non-perturbative procedure for the surface reconstruction. This task requires solving an additional nonlinear PDE (i.e., Eq. 5.9), but the computation is relatively straightforward and is not necessarily performed at each instant t (only when data on the free surface are needed). Moreover, because this PDE is solved over a short interval $s \in [-1, 0)$, the associated cost is insignificant. The same numerical methods as described above for (5.6) are applied to solving (5.9) in space and time, with the same spatial and temporal resolutions. In particular, the same step size $\Delta s = \Delta t$ is specified for the integration over s.

As part of the numerical tests, we will compare predictions from (5.6) to those from (5.4) (with truncated linear dispersion) and from the full water wave system (2.3). These models are discretized similarly to (5.6). Following Craig and

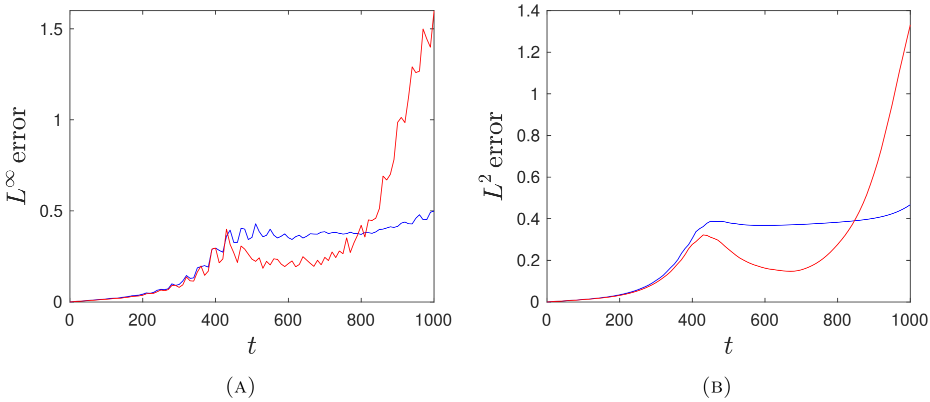


FIGURE 3. Relative errors on η between the fully and weakly nonlinear solutions for $A_0 = 0.02$, $k_0 = 5$, $\lambda_0 = 1$. Blue: Hamiltonian Dysthe with exact dispersion; red: Hamiltonian Dysthe with truncated dispersion.

Sulem [12], the DNO in (2.3) is approximated via the series expansion (2.4) in light of its analyticity properties, which implies that a small number m of terms is sufficient to achieve highly accurate results. The value $m = 6$ is selected based on previous extensive trials [26, 40].

6.3. **Numerical results.** In our simulations of (5.4) and (5.6), the initial condition is a perturbed Stokes wave defined by

$$u(x, 0) = B_0\big[1 + 0.1\cos(\lambda_0 x)\big]. \tag{6.2}$$

We examine in more detail the two cases $(A_0, k_0) = (0.02, 5)$ and $(0.01, 10)$ previously illustrated in the stability analysis of Section 6.1. The length of the computational domain is set to $L = 2\pi$, with spatial and temporal resolutions $\Delta x = 0.012$ ($N = 512$) and $\Delta t = 0.005$ respectively. We first consider the case of "smooth" initial data with $A_0 = 0.01$, $k_0 = 5$ and $\lambda_0 = 1$. A measure of the initial wave steepness is given by $\varepsilon \sim k_0 A_0 = 0.1$ but we should keep in mind that the local steepness will likely increase during the wave evolution as a result of modulational instability. Figure 2 shows the comparison on η between predictions from (2.3), (5.4) and (5.6) at various instants up to $t = 1000$. To allow for a meaningful comparison, the full system (2.3) is initialized by the numerical solution (η, ξ) of (3.10) at $s = -1$, via initial conditions at $s = 0$ provided by (3.2) and (5.10) together with (6.2). Accordingly, all three solutions coincide at $t = 0$ as confirmed by Figure 2(A). The modulational instability is clearly visible, displaying maximum growth around $t = 370$ under excitation of the first sideband mode in agreement with the previous stability analysis. A near-recurrent behavior where the wavetrain undergoes a cycle of modulation and demodulation seems to take place over

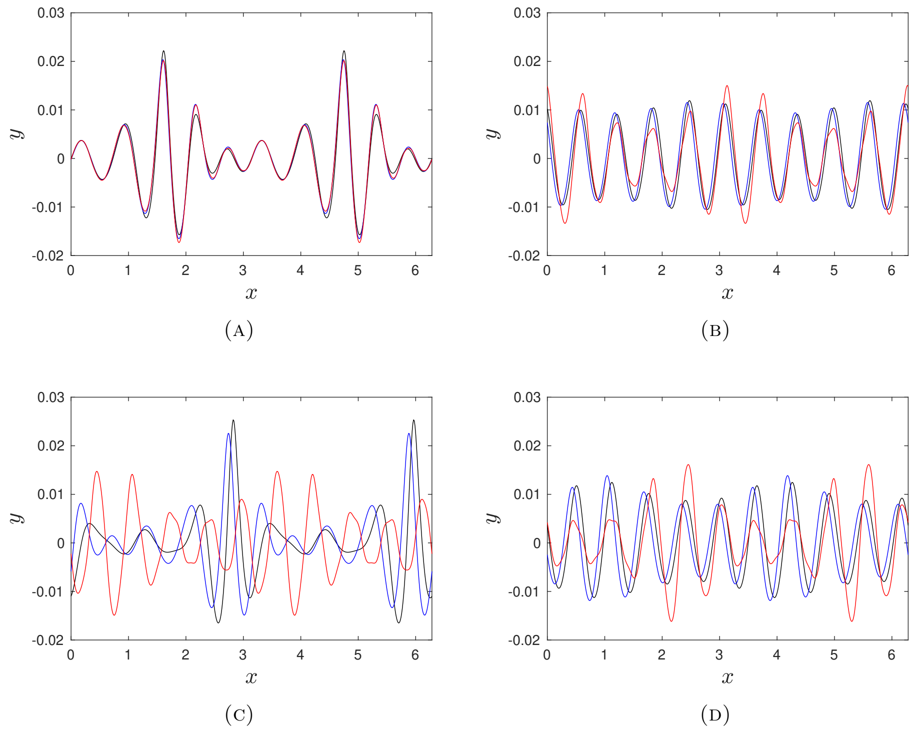


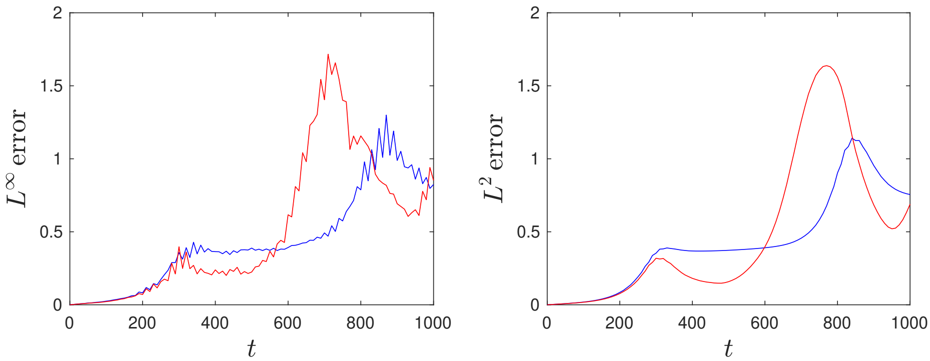
FIGURE 4. Comparison on η at (A) $t = 240$, (B) $t = 590$, (C) $t = 820$, (D) $t = 1000$ for $A_0 = 0.01$, $k_0 = 10$, $\lambda_0 = 2$. Black: full water wave system; blue: Hamiltonian Dysthe with exact dispersion; red: Hamiltonian Dysthe with truncated dispersion.

this time interval. As expected, discrepancies become more pronounced over time, with the exact-dispersion Dysthe solution slightly trailing behind the fully nonlinear solution, while the truncated-dispersion one has deviated much further. These discrepancies are quite apparent at the late time $t = 1000$.

A more quantitative assessment is provided in Figure 3 which plots the time evolution of the relative L^∞ and L^2 errors

$$\frac{\|\eta_f - \eta_w\|_\infty}{\|\eta_f\|_\infty} \quad \text{and} \quad \frac{\|\eta_f - \eta_w\|_2}{\|\eta_f\|_2} \tag{6.3}$$

on η between the fully (η_f) and weakly (η_w) nonlinear solutions. Although the exact-dispersion Dysthe solution seems to underperform near $t = 450$, the corresponding errors then tend to stabilize around 40% afterwards. By contrast, the truncated-dispersion Dysthe solution experiences a sharp loss of accuracy past $t = 700$. The larger errors near $t = 450$ in the exact-dispersion dynamics may be attributed to imbalance in the truncation order for this model (exact linear



(A) (B)

FIGURE 5. Relative errors on η between the fully and weakly nonlinear solutions for $A_0 = 0.01$, $k_0 = 10$, $\lambda_0 = 2$. Blue: Hamiltonian Dysthe with exact dispersion; red: Hamiltonian Dysthe with truncated dispersion.

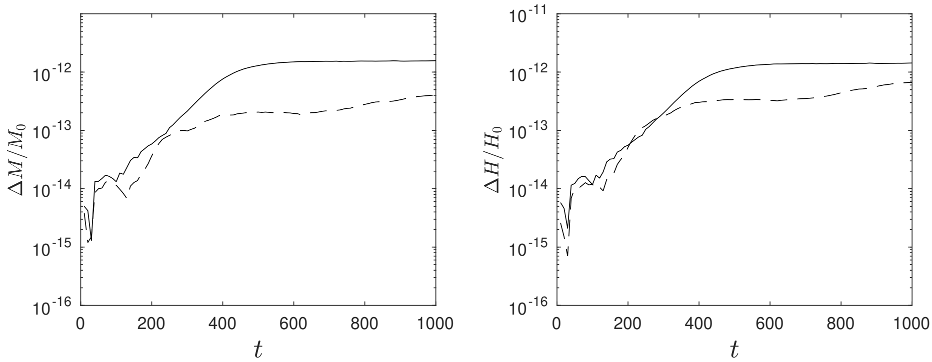
terms vs. approximate nonlinear terms) which would be felt most severely around the time of maximum instability.

We now turn our attention to the case of "rougher" initial data with $A_0 = 0.01$, $k_0 = 10$ and $\lambda_0 = 2$ (Figure 4). As expected, this regime is more prone to modulational instability, and maximum growth is reached sooner around $t = 240$. Consistent with the previous stability analysis, the second sideband mode tends to develop here. A near-recurrent behavior is also observed, with two cycles of modulation-demodulation taking place during the time interval $t \in [0, 1000]$. We see again that the exact-dispersion Dysthe solution remains close to the fully nonlinear one, while the truncated-dispersion Dysthe model shows larger discrepancies in both amplitude and phase. These results are mirrored on the plots of the L^∞ and L^2 errors, as depicted in Figure 5. Both Dysthe equations produce a significant peak in the errors around $t = 800$ which corresponds to maximum growth during the second cycle of modulation-demodulation. The error peak associated with the truncated-dispersion Dysthe solution is particularly large and, for either model, it is an indication that the Dysthe approximation may no longer be valid at such an advanced stage of wave evolution.

Finally, the time evolution of the relative errors

$$\frac{\Delta M}{M_0} = \frac{|M - M_0|}{M_0} \quad \text{and} \quad \frac{\Delta H}{H_0} = \frac{|H - H_0|}{H_0}$$

on wave action (5.5) and energy (5.7) associated with (5.6) is illustrated in Figure 6 for $(A_0, k_0, \lambda_0) = (0.02, 5, 1)$ and $(0.01, 10, 2)$. Integrals in the definition of these quantities and in the L^2 norm (6.3) are computed via the trapezoidal rule over the periodic cell $[0, L]$. The reference values M_0 and H_0 denote the initial values



(A) (B)

FIGURE 6. Relative errors on M and H for the Hamiltonian Dysthe equation (5.6) with exact dispersion. Solid line: $(A_0, k_0, \lambda_0) = (0.02, 5, 1)$; dashed line: $(A_0, k_0, \lambda_0) = (0.01, 10, 2)$.

of M and H at $t = 0$. Overall, these quantities are very well conserved in both parameter regimes. After a transitional increase until about $t = 500$, their errors seem to stabilize around 10^{-13}–10^{-12} from this point on.

7. Conclusions

We present a detailed review of the Hamiltonian modulational approach that has been recently proposed in [9, 13, 10, 24] and which has been applied to deriving Hamiltonian versions of Dysthe's equation (a higher-order NLS model) for two- and three-dimensional gravity waves on deep water. We describe the main steps in the derivation of the evolution equation for the complex wave envelope, together with the procedure to reconstruct the free surface from this wave envelope. The methodology consists in performing a sequence of canonical transformations to the water wave system expressed in Zakharov's Hamiltonian formulation, and involves various asymptotic scalings, a modulational Ansatz and a homogenization lemma. A central role is played by the Birkhoff normal form transformation to eliminate all non-resonant cubic terms, which leads to a simple and elegant form of the reduced Hamiltonian. In addition, this normal form transformation yields a non-perturbative procedure to reconstruct the free surface via the solution of an auxiliary Hamiltonian system. For the two-dimensional problem, this procedure requires solving an inviscid Burgers equation. All these calculations are performed within the Hamiltonian framework, consistent with Zakharov's original viewpoint, and are facilitated by the use of an explicit series expansion of the Dirichlet–Neumann operator associated with the fluid domain.

To illustrate the performance of this weakly nonlinear approximation, we show numerical simulations of the Hamiltonian Dysthe equation using the version with the exact linear dispersion relation. We compare them to computations from the full water wave system in the context of modulational instability of Stokes waves and find very good agreement.

It would be of interest to extend these results to the finite-depth case in the two- and three-dimensional setting. Preliminary studies in this direction can be found in [14, 11, 9]. These problems pose additional challenges and are envisioned for future work.

References

[1] M. J. Ablowitz and H. Segur, On the evolution of packets of water waves, *J. Fluid Mech.* **92** (1979), no. 4, 691–715. MR 0544892.

[2] M. Berti, R. Feola, and F. Pusateri, Birkhoff normal form and long time existence for periodic gravity water waves, *Comm. Pure Appl. Math.*, early view, 2022. https://doi.org/10.1002/cpa.22041.

[3] M. Berti, R. Feola, and F. Pusateri, Birkhoff normal form for gravity water waves, *Water Waves* **3** (2021), no. 1, 117–126. MR 4246390.

[4] U. Brinch-Nielsen and I. G. Jonsson, Fourth order evolution equations and stability analysis for Stokes waves on arbitrary water depth, *Wave Motion* **8** (1986), no. 5, 455–472. MR 0858522.

[5] H. Chihara, *Third order semilinear dispersive equations related to deep water waves*, arXiv:math/0404005 [math.AP], 2004.

[6] R. R. Coifman and Y. Meyer, Nonlinear harmonic analysis and analytic dependence, in *Pseudodifferential Operators and Applications (Notre Dame, Ind., 1984)*, 71–78, Proc. Sympos. Pure Math., 43, Amer. Math. Soc., Providence, RI, 1985. MR 0812284.

[7] W. Craig, P. Guyenne, and H. Kalisch, Hamiltonian long-wave expansions for free surfaces and interfaces, *Comm. Pure Appl. Math.* **58** (2005), no. 12, 1587–1641. MR 2177163.

[8] W. Craig, P. Guyenne, D. P. Nicholls, and C. Sulem, Hamiltonian long-wave expansions for water waves over a rough bottom, *Proc. R. Soc. Lond. Ser. A* **461** (2005), no. 2055, 839–873. MR 2121939.

[9] W. Craig, P. Guyenne, and C. Sulem, A Hamiltonian approach to nonlinear modulation of surface water waves, *Wave Motion* **47** (2010), no. 8, 552–563. MR 2734546.

[10] W. Craig, P. Guyenne, and C. Sulem, Normal form transformations and Dysthe's equation for the nonlinear modulation of deep-water gravity waves, *Water Waves* **3** (2021), no. 1, 127–152. MR 4246391.

[11] W. Craig, U. Schanz, and C. Sulem, The modulational regime of three-dimensional water waves and the Davey-Stewartson system, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), no. 5, 615–667. MR 1470784.

[12] W. Craig and C. Sulem, Numerical simulation of gravity waves, *J. Comput. Phys.* **108** (1993), no. 1, 73–83. MR 1239970.

[13] W. Craig and C. Sulem, Mapping properties of normal forms transformations for water waves, *Boll. Unione Mat. Ital.* **9** (2016), no. 2, 289–318. MR 3502161.

[14] W. Craig, C. Sulem, and P.-L. Sulem, Nonlinear modulation of gravity waves: a rigorous approach, *Nonlinearity* **5** (1992), no. 2, 497–522. MR 1158383.

[15] W. Craig and P. A. Worfolk, An integrable normal form for water waves in infinite depth, *Phys. D* **84** (1995), no. 3-4, 513–531. MR 1336546.

[16] A. Davey and K. Stewartson, On three-dimensional packets of surface waves, *Proc. R. Soc. Lond. Ser. A* **338** (1974), 101–110. MR 0349126.

[17] A. I. Dyachenko, D. I. Kachulin, and V. E. Zakharov, Super compact equation for water waves, *J. Fluid Mech.* **828** (2017), 661–679. MR 3707778.

[18] A. I. Dyachenko and V. E. Zakharov, Is free-surface hydrodynamics an integrable system?, *Phys. Lett. A* **190** (1994), no. 2, 144–148. MR 1283779.

[19] A. I. Dyachenko and V. E. Zakharov, Compact equation for gravity waves on deep water, *JETP Lett.* **93** (2011), 701–705. https://doi.org/10.1134/S0021364011120058.

[20] K. B. Dysthe, Note on a modification to the nonlinear Schrödinger equation for application to deep water waves, *Proc. R. Soc. Lond. Ser. A* **369** (1979), 105–114. https://doi.org/10.1098/rspa.1979.0154.

[21] F. Fedele and D. S. Dutykh, Hamiltonian form and solitary waves of the spatial Dysthe equations, *Pis'ma v Zh. Èksper. Teoret. Fiz.* **94** (2011), no. 12, 921–925; *JETP Lett.* **94** (2011), no. 12, 840–844. https://www.mathnet.ru/eng/jetpl2402.

[22] O. Gramstad and K. Trulsen, Hamiltonian form of the modified nonlinear Schrödinger equation for gravity waves on arbitrary depth, *J. Fluid Mech.* **670** (2011), 404–426. MR 2773687.

[23] R. Grande, K. M. Kurianski, and G. Staffilani, On the nonlinear Dysthe equation, *Nonlinear Anal.* **207** (2021), Paper No. 112292, 36 pp. MR 4220762.

[24] P. Guyenne, A. Kairzhan, and C. Sulem, Hamiltonian Dysthe equation for three-dimensional deep-water gravity waves, *Multiscale Model. Simul.* **20** (2022), no. 1, 349–378. MR 4395161.

[25] P. Guyenne, A. Kairzhan, C. Sulem, and B. Xu, Spatial form of a Hamiltonian Dysthe equation for deep-water gravity waves, *Fluids* **6** (2021), no. 3, 103. https://doi.org/10.3390/fluids6030103.

[26] P. Guyenne and D. P. Nicholls, A high-order spectral method for nonlinear water waves over moving bottom topography, *SIAM J. Sci. Comput.* **30** (2007/08), no. 1, 81–101. MR 2377432.

[27] H. Hasimoto and H. Ono, Nonlinear modulation of gravity waves, *J. Phys. Soc. Japan* **33** (1972), 805–811. https://doi.org/10.1143/JPSJ.33.805.

[28] S. J. Hogan, The fourth-order evolution equation for deep-water gravity-capillary waves, *Proc. R. Soc. Lond. Ser. A* **402** (1985), 359–372. https://doi.org/10.1098/rspa.1985.0122.

[29] H. Koch and J.-C. Saut, Local smoothing and local solvability for third order dispersive equations, *SIAM J. Math. Anal.* **38** (2006/07), no. 5, 1528–1541. MR 2286018.

[30] V. P. Krasitskii, On reduced equations in the Hamiltonian theory of weakly nonlinear surface waves, *J. Fluid Mech.* **272** (1994), 1–20. MR 1289106.

[31] E. Lo and C. C. Mei, A numerical study of water-wave modulation based on a higher-order nonlinear Schrödinger equation, *J. Fluid Mech.* **150** (1985), 395–416. https://doi.org/10.1017/S0022112085000180.

[32] R. Mosincat, D. Pilod, and J.-C. Saut, Global well-posedness and scattering for the Dysthe equation in $L^2(\mathbb{R}^2)$, *J. Math. Pures Appl. (9)* **149** (2021), 73–97. MR 4238997.

[33] S. Nazarenko, *Wave Turbulence*, Lecture Notes in Physics, 825, Springer, Heidelberg, 2011. MR 3014432.

[34] L. Shemer, E. Kit, and H.-Y. Jiao, An experimental and numerical study of the spatial evolution of unidirectional nonlinear water-wave groups. *Phys. Fluids* **14** (2002), 3380–3390. https://doi.org/10.1063/1.1501908

[35] C. Sulem and P.-L. Sulem, *The Nonlinear Schrödinger Equation*, Applied Mathematical Sciences, 139, Springer-Verlag, New York, 1999. MR 1696311.

[36] K. Trulsen, Weakly nonlinear sea surface waves—freak waves and deterministic forecasting, in *Geometric Modelling, Numerical Simulation, and Optimization: Applied Mathematics at SINTEF*, 191–209, Springer, Berlin, 2007. MR 2348923.

[37] K. Trulsen and K. B. Dysthe, A modified nonlinear Schrödinger equation for broader bandwidth gravity waves on deep water, *Wave Motion* **24** (1996), no. 3, 281–289. MR 1419980.

[38] K. Trulsen, I. Kliakhandler, K. B. Dysthe, and M. G. Velarde, On weakly nonlinear modulation of waves on deep water, *Phys. Fluids* **12** (2000), no. 10, 2432–2437. MR 1789997.

[39] S. Wu, *The quartic integrability and long-time existence of steep water waves in 2D*, arXiv:2010.09117v2 [math.AP], 2021.

[40] L. Xu and P. Guyenne, Numerical simulation of three-dimensional nonlinear water waves, *J. Comput. Phys.* **228** (2009), no. 22, 8446–8466. MR 2574099.

[41] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.* **9** (1968), 190–194. https://doi.org/10.1007/BF00913182.

[42] V. Zakharov, Statistical theory of gravity and capillary waves on the surface of a finite-depth fluid, *Eur. J. Mech. B Fluids* **18** (1999), no. 3, 327–344. MR 1701696.

P. Guyenne
Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA
guyenne@udel.edu

A. Kairzhan
Department of Mathematics, University of Toronto, Toronto, Ontario M5S 2E4, Canada
kairzhan@math.toronto.edu

C. Sulem ✉
Department of Mathematics, University of Toronto, Toronto, Ontario M5S 2E4, Canada
sulem@math.utoronto.ca

Received: October 4, 2021
Accepted: February 5, 2022