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POSITIVITIES IN HALL–LITTLEWOOD EXPANSIONS AND RELATED PLETHYSTIC OPERATORS

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Abstract. The Hall–Littlewood polynomials $\mathbf{H}_\lambda = Q'_\lambda[X;q]$ are an important symmetric function basis that appears in many contexts. In this work, we give an accessible combinatorial formula for expanding the related symmetric functions $\mathbf{H}_\alpha$ for any composition α, in terms of the complete homogeneous basis. We do this by analyzing Jing's operators, which extend to give nice expansions for the related symmetric functions $\mathbf{C}_\alpha$ and $\mathbf{B}_\alpha$ which appear in the formulation of the Compositional Shuffle Theorem. We end with some consequences related to eigenoperators of the modified Macdonald basis.

1. Introduction

After Macdonald introduced his Macdonald polynomials and his positivity conjectures [37], Garsia and Haiman introduced their modified version $\{\widetilde{\mathrm{H}}_\mu[X;q,t]\}_\mu$, giving Macdonald polynomials a representation theoretical setting by stating that $\widetilde{\mathrm{H}}_\mu$ is the Frobenius characteristic of the bigraded S_n-module now referred to as the Garsia–Haiman module, M_μ [17, 15]. These modules are defined by starting with an alternant

$$\Delta_\mu = \det \|x_i^s y_i^r\|_{\substack{(r,s)\in\mu \\ i=1,\dots,n}}$$

and setting $M_\mu = \{P(\partial)\Delta_\mu : P \in \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]\}$ to be the linear span of derivatives of Δ_μ, with S_n acting diagonally on both sets of variables. The Frobenius characteristic sends an irreducible representation of S_n indexed by a partition $\lambda \vdash n$ to the Schur function s_λ, and the grading given by the variables $x_1,\dots,x_n$ and $y_1,\dots,y_n$ is recorded by the powers of t and q respectively. This would give

$$\mathcal{F}M_\mu = \sum_{\lambda\vdash n} \widetilde{\mathrm{K}}_{\lambda,\mu}(q,t)s_\lambda,$$

with $\widetilde{\mathrm{K}}_\lambda \in \mathbb{N}[q,t]$, since the coefficients of q and t give multiplicities of irreducible representations. The assertion that $\mathcal{F}M_\mu = \widetilde{\mathrm{H}}_\mu$, then proved by Haiman [28], would show that the modified Macdonald polynomials have a positive expansion in terms of the Schur basis, proving the positivity element of Macdonald's conjectures.

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Haiman's proof of this conjecture was done by geometrical means, finding a connection between Macdonald polynomials and the Hilbert scheme of points on the plane $\mathbb{C}^2$. Haiman also proves more through this association [29]: Let R_n denote the quotient of $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ by the ideal generated by the S_n invariants with zero constant term. Since the invariants are generated by the polarized power sum polynomials

$$\sum_{i=1}^{n} x_i^r y_i^s \qquad \text{with } r+s>0,$$

one can equivalently study the space of diagonal harmonics

$$DH_n = \Big\{ P \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] : \sum_i \partial_{x_i}^r \partial_{y_i}^s P = 0 \text{ for } r+s>0 \Big\}.$$

Haiman proves that

$$\mathcal{F} R_n = \mathcal{F} DH_n = \nabla e_n,$$

where ∇ is the eigenoperator of the modified Macdonald basis defined in [4], given by $\nabla \widetilde{\mathrm{H}}_\mu = T_\mu \widetilde{\mathrm{H}}_\mu$ with $T_\mu = t^{n(\mu)} q^{n(\mu')}$.

It is important to note that ever since Haiman's proof, a growing number of results and conjectures have given Macdonald polynomials and their eigenoperators a presence in several areas of study. For one, it was shown by Hogancamp [31] that $\nabla^m e_n$ produces the triply graded knot invariant for the $(n, nm+1)$ torus knot. There is also the work of Mellit [40, 38], which gives the invariant for the (m, n) torus knot in terms of the related operators $Q_{m,n}$ on symmetric functions. Gorsky, Neguţ, and Rasmussen have conjectured an association between a certain derived category of equivariant sheaves over the flag Hilbert scheme and complexes of Soergel bimodules [20]. The applications and presence of Macdonald polynomials are far-reaching. However, in this work, we will mainly be concerned with operators which appear in the statement of the Compositional Shuffle Theorem.

The Shuffle Theorem, conjectured in [23], gives a combinatorial expansion for ∇e_n in terms of labeled Dyck paths or parking functions. This will be elaborated in Section 7. Haglund, Morse, and Zabrocki [24] later found a compositional refinement of this conjecture by introducing a family of operators $\{C_a\}_{a \in \mathbb{Z}}$, which we introduce in Section 3. They show that $\sum_{\alpha \models n} \mathbf{C}_\alpha = e_n$, and they conjecture that $\nabla \mathbf{C}_\alpha$ can be expressed as a sum over labeled Dyck paths which return to the diagonal according to the composition α. Carlsson and Mellit proved this compositional refinement, and thus the Shuffle Theorem, by introducing a new family of operators called the Dyck path algebra [8].

This story describes three aspects which often appear in this area of study. On the one hand, there is a symmetric function, often associated to plethystic operators or eigenoperators of the modified Macdonald basis. There is then the question of giving a combinatorial expansion of this symmetric function in terms of a classical basis. The last aspect is to show that the symmetric function is the Frobenius characteristic of some natural S_n-module. Section 7 will describe how this story is mirrored for delta eigenoperators in the context of the Delta Theorem, conjectured in [25]. The compositional refinement of the Delta Theorem, conjectured in [10], was recently proved by D'Adderio and Mellit [11]. There is also the proof of the Extended Delta Theorem in [6]. Section 6 will also describe an analogous story for Hall–Littlewood polynomials, which will be the main focus of our work.

Bergeron noted that many of the symmetric functions which appear in the theory of modified Macdonald polynomials, its related eigenoperators, and those related to triply graded knot invariants have a remarkable positivity property. Namely, that they become positive in terms of the elementary or homogeneous symmetric function bases by simply substituting q to be $1+u$ [3]. More precisely, we will say that a symmetric function $F[X;q,t]$ exhibits the e-positivity phenomenon if when expanded in terms of the elementary basis, we have

$$F[X;1+u,t] = \sum_{\lambda} c_{\lambda}(u,t) e_{\lambda}$$

with $c_{\lambda}(u,t) \in \mathbb{N}[u,t]$. Similarly, we may also say that a symmetric function exhibits the h-positivity phenomenon if the same holds true when expanded in terms of the homogeneous basis. For instance, $\nabla e_n|_{q\to 1+u}$ is e-positive by the positivity of Column (or Vertical Strip) LLT polynomials proved by D'Adderio [9], together with the proof of the Shuffle Theorem by Carlsson and Mellit. For an explicit combinatorial expansion, one may use the combinatorial formula for Column LLT polynomials proved by Alexandersson and Sulzgruber [2]. This connection to LLT polynomials will be further explained in Section 7.

Here, we will be concerned with looking at plethystic operators on symmetric functions that generate important families of symmetric functions. The main operators we will study are Jing's operators $\{H_k\}_k$, which generate Hall–Littlewood polynomials [33]. Haglund, Morse, and Zabrocki introduced modified versions, the B and C operators [24], which play a vital role in the statement and proof of the compositional Shuffle Theorem. Similar positivities and expansions can also be proved for these operators. In Theorem 5.2, we will give a combinatorial expansion of Hall–Littlewood related symmetric functions $\mathbf{H}_{\alpha}$ indexed by a composition α in terms of the homogeneous basis. This is done by analyzing Jing's operators which generate $\mathbf{H}_{\alpha}$. In particular, we will see that $\mathbf{H}_{\alpha}$ exhibits the h-positivity phenomenon, and its homogeneous expansion can be given directly by a new combinatorial formula.

Hall–Littlewood symmetric functions are important in their own right, for which we include more details in Section 6. They can be produced by taking the Frobenius characteristic of the cohomology ring of the Springer fibers, as is seen in the work of Hotta and Springer [32]. Lascoux and Schützenberger gave a combinatorial formula for the Schur expansion in terms of the charge statistic [36], with a complete proof given in the work of Butler [7]. And more recently, Mellit proved an explicit formula involving Macdonald polynomials for the Poincaré polynomials of parabolic character varieties of Riemann surfaces with semisimple local monodromies [39]. Here, Mellit shows that one can get the modified Macdonald polynomial from the Hall–Littlewood polynomials. We will see that there are some remarkable underlying properties and questions that warrant further study.

2. Symmetric functions and plethystic substitution

As a reference for symmetric functions, we have [37]. For plethystic substitution and some of the operators defined here, it may be useful to cite [18] and [5].

We start by recalling the usual classical bases of symmetric functions: the power sum $\{p_\lambda\}_\lambda$, (complete) homogeneous $\{h_\lambda\}_\lambda$, elementary $\{e_\lambda\}_\lambda$, monomial $\{m_\lambda\}_\lambda$ and Schur basis $\{s_\lambda\}_\lambda$. The power, homogeneous and elementary bases are multiplicative in the sense that, for any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)})$,

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}, \qquad h_\lambda = h_{\lambda_1} \cdots h_{\lambda_{\ell(\lambda)}}, \quad \text{and} \qquad e_\lambda = e_{\lambda_1} \cdots e_{\lambda_{\ell(\lambda)}}.$$

The *Hall scalar product* is given by setting

$$\langle p_\lambda, p_\mu \rangle = z_\mu \chi(\lambda = \mu),$$

where $\chi(A)$ is the indicator function giving 1 if A is true and 0 otherwise; and $z_\mu = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ when the multiplicity of i in μ is given by m_i. The Hall scalar product gives

$$\langle s_\lambda, s_\mu \rangle = \langle h_\lambda, m_\mu \rangle = \chi(\lambda = \mu).$$

The *adjoint* $F^\perp$ of a symmetric function F is defined as the operator which gives

$$\langle F^\perp G, H \rangle = \langle G, FH \rangle$$

for all symmetric functions G and H.

For any expression $E(t_1, t_2, \dots)$ in the variables $t_1, t_2, \dots$, define

$$p_k \left[E(t_1, t_2, \dots) \right] = E(t_1^k, t_2^k, \dots) \qquad \text{and} \qquad p_\lambda[E] = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[E].$$

Any symmetric function F can be expanded in terms of the power basis to give

$$F = \sum_\lambda c_\lambda p_\lambda$$

for some scalar coefficients c_λ. The *plethystic substitution* of F at E can then be defined by setting

$$F[E] = \sum_\lambda c_\lambda p_\lambda[E].$$

For $X = x_1 + x_2 + \cdots$, we have that

$$p_k[X] = x_1^k + x_2^k + \cdots = p_k$$

is the usual power sum symmetric function of degree k. We also have that

$$p_k[-X] = -(x_1^k + x_2^k + \cdots) = -p_k,$$

meaning that substituting negative variables within the plethystic bracket is not the same as the usual variable substitution. A useful device is the introduction of the variable ϵ which behaves as a variable within the plethystic bracket and is then evaluated to -1 outside the bracket. With this convention, we have

$$p_k[\epsilon X] = (-1)^k p_k[X] \qquad \text{and} \qquad p_k[-\epsilon X] = (-1)^{k-1} p_k[X].$$

It follows that the classical involution ω, which sends p_k to $(-1)^{k-1}p_k$, e_k to h_k, and s_λ to $s_{\lambda'}$ (where λ' is the conjugate partition of λ), is given by

$$\omega F[X] = F[-\epsilon X].$$

We also have, for instance, that $h_k[-X] = h_k[-\epsilon\epsilon X] = e_k[\epsilon X] = (-1)^k e_k[X]$.

We now introduce the fundamental translation and multiplication operators, defined for any expression Y, by setting

$$\mathcal{T}_Y\, F[X] = F[X+Y] \qquad \text{and} \qquad \mathcal{P}_Y\, F[X] = \operatorname{Exp}[XY]F[X],$$

where

$$\operatorname{Exp}[X] = \exp\left(\sum_{k\geq 1} \frac{p_k}{k}\right) = \sum_{n\geq 0} h_n$$

is the usual generating series of homogeneous symmetric functions. The importance of these operators is given by the following proposition.

Proposition 2.1 ([18, Theorem 1.1])**.** *For any expression Y, we have the following equalities of operators:*

$$\mathcal{T}_Y = \sum_\lambda s_\lambda[Y] s_\lambda^\perp \qquad \textit{and} \qquad \mathcal{P}_Y\, F[X] = \sum_\lambda s_\lambda[Y] \underline{s}_\lambda[X].$$

Here, $s_\lambda^\perp$ is the adjoint of the multiplication operator $\underline{s}_\lambda[X]$ under the Hall scalar product.

This proposition can be proved by simply looking at the case when $Y = y_1 + y_2 + \cdots$ and applying $\mathcal{T}_Y$ and $\mathcal{P}_Y$ to a Schur function $s_\mu[X]$. In the first case we have

$$\mathcal{T}_Y\, s_\mu[X] = s_\mu[X+Y].$$

We now fill the shape μ by $y_1, y_2, \ldots, x_1, x_2, \ldots$ in a semistandard fashion. This means that the variables $y_1, y_2, \ldots$ have to fill some subshape of μ, say λ, and the remaining portion of μ with λ removed, denoted μ/λ, must be filled by x_i's. We get

$$\begin{aligned} s_\mu[X+Y] &= \sum_{\lambda\subseteq\mu} s_\lambda[Y] s_{\mu/\lambda}[X] = \sum_{\lambda\subseteq\mu} s_\lambda[Y]\left(s_\lambda^\perp s_\mu\right)[X] \\ &= \left(\sum_{\lambda\subseteq\mu} s_\lambda[Y] s_\lambda^\perp\right) s_\mu[X]. \end{aligned}$$

The important fact here, which we will use later in this paper, is that $s_\lambda^\perp s_\mu = s_{\mu/\lambda}$. Since $\{s_\mu\}_\mu$ is a basis for symmetric functions, we get an equality of operators.

The second identity of the proposition follows from the Cauchy identity, which states that, for any two homogeneous bases $\{u_\lambda\}_\lambda$ and $\{v_\lambda\}_\lambda$ which are dual under the Hall scalar product, we have

$$\operatorname{Exp}[XY] = \sum_\lambda u_\lambda[X] v_\lambda[Y].$$

Taking $u_\lambda = v_\lambda = s_\lambda$ we obtain the second equality of operators.

If $Y = z$ is a single variable, then we have

$$s_\lambda[z] = \begin{cases} 0 & \text{if } \ell(\lambda) > 1, \text{ and} \\ z^{|\lambda|} & \text{otherwise.} \end{cases}$$

This gives the following specializations of the translation and multiplication operators:

$$\mathcal{T}_z = \sum_{n\geq 0} z^n h_n^\perp \qquad \text{and} \qquad \mathcal{P}_z = \sum_{n \geq 0} z^n \underline{h}_n[X].$$

Similarly, we have

$$\mathcal{T}_{-z} = \sum_{n\geq 0} (-z)^n e_n^\perp \qquad \text{and} \qquad \mathcal{P}_{-z} = \sum_{n \geq 0} (-z)^n \underline{e}_n[X].$$

3. Plethystic operators

Of importance to us are certain plethystic operators that play a role in generating important families of symmetric functions. We start with Jing's operators which generate the Hall–Littlewood symmetric functions [33]. They can be defined plethystically, as was done by Garsia and Procesi in [19], by setting

$$H(z) = \sum_k z^k H_k = \mathcal{P}_z \mathcal{T}_{\frac{q-1}{z}}.$$

Using the proposition from the previous section, we can rewrite this operator. First note by the definition of the translation operators that we can write $\mathcal{T}_{Y+Z} = \mathcal{T}_Y \mathcal{T}_Z$. Then

$$\mathcal{P}_z \mathcal{T}_{\frac{q-1}{z}} = \mathcal{P}_z \mathcal{T}_{\frac{q}{z}} \mathcal{T}_{\frac{-1}{z}} = \sum_{n \geq 0} z^n \underline{h}_n \sum_{a \geq 0} \left(\frac{q}{z}\right)^a h_a^\perp \sum_{b\geq 0} \left(\frac{-1}{z}\right)^b e_b^\perp.$$

It follows that

$$H_k = \sum_{a,b\geq 0} \underline{h}_{a+b+k} q^a (-1)^b h_a^\perp e_b^\perp,$$

where, again, for any symmetric function F, we use the notation $\underline{F}$ to mean the operation of multiplication by F. For any sequence of positive integers $\alpha = (\alpha_1, \dots, \alpha_\ell)$, let $\mathbf{H}_\alpha = H_{\alpha_1} \cdots H_{\alpha_\ell}(1)$. Jing proves the following proposition.

Proposition 3.1 ([33])**.** *For any partition λ, $\mathbf{H}_\lambda$ is the Hall–Littlewood symmetric function, often denoted by $Q'_\lambda[X;q]$, defined by the relations*

$$\langle Q'_\lambda[X(1-q)], Q'_\mu[X]\rangle = 0 \quad \textit{for } \lambda \neq \mu \qquad \textit{and} \qquad \langle Q'_\lambda[X], h_n\rangle = q^{n(\lambda)}.$$

Another important symmetric function operator is attained by setting $B_k = \omega H_k \omega^{-1}$. To see how ω commutes with translation or multiplication operators, we start by noting that

$$\omega\,\mathcal{T}_Y\,\omega s_\mu = \omega\,\mathcal{T}_Y\,s_\mu[-\epsilon X] = \omega s_\mu[-\epsilon(X+Y)] = s_\mu[X-\epsilon Y] = \mathcal{T}_{-\epsilon Y}\,s_\mu[X].$$

Similarly, we also have

$$\omega\,\mathcal{P}_Y\,\omega s_\mu[X] = \omega\,\mathrm{Exp}[XY]s_\mu[-\epsilon X] = \mathrm{Exp}[-\epsilon XY]s_\mu[X] = \mathcal{P}_{-\epsilon Y}\,s_\mu[X].$$

Therefore, if we set $B(z) = \omega H(z)\omega$, then we have

$$B(z) = \omega\,\mathcal{P}_z\,\omega\omega\,\mathcal{T}_{\frac{q-1}{z}}\,\omega = \mathcal{P}_{-\epsilon z}\,\mathcal{T}_{-\epsilon\frac{q-1}{z}}\,.$$

We can then write

$$\begin{aligned} B_k = \mathcal{P}_{-\epsilon z}\,\mathcal{T}_{\epsilon\frac{1-q}{z}}\Big|_{z^k} &= \mathcal{P}_{-\epsilon z}\,\mathcal{T}_{\frac{\epsilon}{z}}\,\mathcal{T}_{\frac{-\epsilon q}{z}}\Big|_{z^k} \\ &= \sum_{a-b-c=k} \underline{e}_a[X](-1)^b h_b^\perp q^c e_c^\perp. \end{aligned}$$

Lastly, there are the C operators, which can be defined by setting

$$C(z) = -q\,\mathcal{P}_{\epsilon\frac{z}{q}}\,\mathcal{T}_{\epsilon\frac{1-q}{z}}\,.$$

The C and B operators play an important role in the Compositional Shuffle Theorem, conjectured in [24] and proved by Carlsson and Mellit in [8]. In particular, some elements of the Dyck path algebra of Carlsson and Mellit can be interpreted using the B operators.

For $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, let $\mathbf{C}_\alpha = C_{\alpha_1}\cdots C_{\alpha_\ell}(1)$ and $\mathbf{B}_\alpha = B_{\alpha_1}\cdots B_{\alpha_\ell}(1)$. Usually, $\mathbf{B}_\alpha$ is defined by writing the operators in the opposite order [24], but in order to simplify our notation, we use this convention. We will show how to directly give combinatorial formulas for these symmetric functions in terms of homogeneous or elementary symmetric functions. Since the computations are similar, we will only give a full analysis of $\mathbf{H}_\alpha = \omega\mathbf{B}_\alpha$. We will begin by analyzing the operator H_k as described in [13]; we will follow by giving a combinatorial formula for $\mathbf{H}_\alpha$ in terms of homogeneous symmetric functions.

4. The Hall–Littlewood operators

We will start by proving the following theorem seen in [13], [9] and [1]. Our proof follows [13], but since it is vital for our final combinatorial expansion, we include the proof for completeness. This combinatorial proof will lead us to the combinatorial expansion of Hall–Littlewood polynomials indexed by compositions, and it gives a general method for giving positivities involving the H, B, and C operators.

Theorem 4.1 ([13, 1])**.** *The family of symmetric functions $\{\mathbf{H}_\alpha\}_{\alpha\models n}$ exhibits the h-positivity phenomenon.*

To show that $\mathbf{H}_\alpha$ exhibits the h-positivity phenomenon, we first prove the following lemma. From here on, we will set $u = q-1$.

Lemma 4.2. *For any partition μ and integer a,*

$$H_a h_\mu = \sum_{\nu \vdash |\mu|+a} h_\nu b_{\mu,\nu}(u)$$

for some $b_{\mu,\nu}(u) \in \mathbb{N}[u]$.

With this given, we can prove the theorem by induction: The base case $H_a(1) = h_a$ is easily verified. Given that for a composition p we have

$$\mathbf{H}_p = \sum_{\mu \vdash |p|} h_\mu c_{p,\mu}(u),$$

we can use the lemma to write

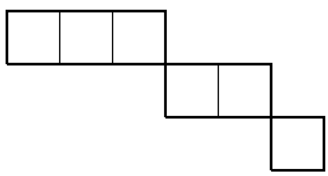
$$\begin{aligned} H_a \mathbf{H}_p &= \sum_{\mu \vdash |p|} H_a h_\mu c_{p,\mu}(u) \\ &= \sum_{\mu \vdash |p|} \sum_{\nu \vdash |\mu|+a} h_\nu b_{\mu,\nu}(u) c_{p,\mu}(u). \end{aligned}$$

This means that if $\mathbf{H}_p$ is h-positive in terms of u, then so is $H_a \mathbf{H}_p$. We now prove the lemma.

Proof. In order to understand the operator H_a, we start by writing

$$H_a = \sum_{r,s} (-1)^s (1+u)^r \underline{h}_{a+r+s} h_r^\perp e_s^\perp,$$

where again, $\underline{h}_k$ is the operator of multiplication by h_k. To find $H_a h_\mu$, we treat h_μ as a skew Schur function. Let $\mu^\times$ be the shape one gets by placing rows of size $\mu_1, \ldots, \mu_\ell$ corner to corner to give a skew diagram with no two cells in the same column. For instance, the shape $(3,2,1)^\times$ is given by the diagram

$(3,2,1)^\times =$ 

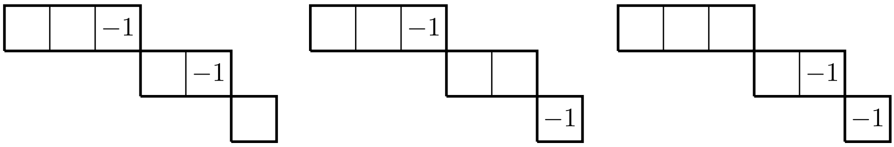
The Pieri rules are then given by removing cells from the rows: For any partition μ of length ℓ, we have

$$h_k^\perp h_\mu = \sum_{\substack{\epsilon_1+\cdots+\epsilon_\ell = k \\ \epsilon_i \geq 0}} \prod_{i=1}^{\ell} h_{\mu_i - \epsilon_i} \qquad \text{and} \qquad e_k^\perp h_\mu = \sum_{\substack{\epsilon_1+\cdots+\epsilon_\ell = k \\ \epsilon_i \in \{0,1\}}} \prod_{i=1}^{\ell} h_{\mu_i - \epsilon_i}.$$

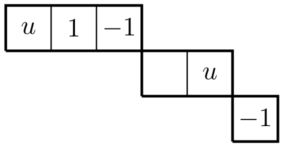
Given r and s, we now construct a set $T_\mu^{r,s}$ of labeled tableaux of shape $\mu^\times$. Each element $T \in T_\mu^{r,s}$ will have a weight $wt(S)$ giving

$$H_a h_\mu = \sum_{r+s \leq |\mu|} h_{a+r+s} \sum_{S \in T_\mu^{r,s}} wt(S).$$

To construct $T_\mu^{r,s}$, first select s rows in $\mu^\times$ and inscribe the rightmost cells with a "-1". For instance, if $s=2$, we have the following three choices for filling $(3,2,1)^\times$:



This describes the effect of applying $(-1)^s e_s^\perp$ to h_μ, giving $(-1)^2 e_2^\perp h_{3,2,1} = h_{2,1,1} + h_{2,2} + h_{3,1}$ in this case. Next choose r cells so that they form a horizontal strip in the remaining shape and choose for each cell whether to inscribe it with a "1" or "u". By horizontal strip, we mean that each selected cell has no unselected cell on its right. One example with $s=2$ and $r=3$ is given by

$T =$  $\in T_{(3,2,1)}^{3,2}$

This describes the effect of applying $(1+u)^r h_r^\perp$ to $e_s^\perp h_\mu$. For $i \in \{1, \dots, \ell(\mu)\}$, let c_i be the number of empty cells in row i of T, and let $\lambda(T)$ be the partition whose parts are given by $c_1, \dots, c_{\ell(\mu)}$ in nondecreasing order. The above example would then produce the partition (1) since there is one empty cell in row 2. Let $\rho(T)$ be the product of the entries in the cells of T. The weight of this object is defined by the product

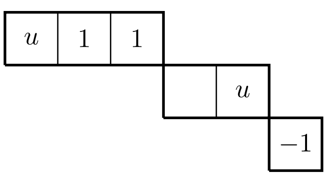
$$wt(S) = \rho(T) \cdot h_{\lambda(S)}.$$

The example above would give $wt(S) = (-1) \cdot u \cdot 1 \cdot u \cdot (-1) \cdot h_1 = u^2 h_1$.

We now show, by using a sign-reversing involution, that

$$\sum_{r+s=k} \sum_{T \in T_\mu^{r,s}} wt(S)$$

is a positive polynomial in u. Given T, scan from left to right for the first cell which is rightmost in its row and is inscribed with either a 1 or a -1. Switch the 1 into a -1 in the first case, and switch the -1 to a 1 in the second case. If no such entry exists, leave the tableau fixed. This is clearly an involution, and it is sign-reversing since we are negating the value of $\rho(S)$ and preserving the number of u's. The above example would be paired with

 $\in T_{(3,2,1)}^{4,1}.$

Let $U_\mu^{r,s}$ be the subset of $T_\mu^{r,s}$ with the condition that if the rightmost cell of a row is labeled, then it contains a u. We have

$$H_a h_\mu = \sum_{r+s\leq|\mu|} h_{a+r+s} \sum_{T\in U_\mu^{r,s}} wt(T),$$

which is a positive polynomial in u, completing our proof of the lemma. □

We can get an actual formula for the application of H_a on a homogeneous element h_μ. Note that the set $U_\mu^{r,s}$ is generated by selecting a certain number r_i of labeled cells in the i^{th} row of $\mu^\times$, filling the first cell with a u, and filling the remaining cells with either a 1 or a u. Therefore, each cell that is not first in its row individually contributes a factor of $1+u=q$. We can write this as follows:

$$H_a h_\mu = \sum_{\substack{r=(r_1,\ldots,r_{\ell(\mu)})\\ 0\leq r_i\leq \mu_i}} h_{a+|r|} \prod_{i=1}^{\ell(\mu)} h_{\mu_i-r_i} \left(uq^{r_i-1}\right)^{\chi(r_i>0)}.$$

Corollary 4.3. *The compositional Hall–Littlewood polynomials $\{\mathbf{H}_\alpha\}_\alpha$ exhibit the h-positivity phenomenon.*

Corollary 4.4. *The family $\{\mathbf{B}_\alpha\}_\alpha$ ranging over compositions α exhibits the e-positivity phenomenon.*

Lastly, we note that

$$\begin{aligned} C_a F[X] &= \left(-\frac{1}{q}\right)^{a-1} F\left[X - \frac{1-1/q}{z}\right] \sum_k z^k h_k\Big|_{z^a} \\ &= (-q)^{a-1}\omega B_a\Big|_{q\to 1/q}\omega F[X]. \end{aligned}$$

This means that

$$\mathbf{C}_\alpha\Big|_{q\to 1/q} = (-q)^{|\alpha|-\ell(\alpha)}\mathbf{H}_\alpha.$$

We therefore get the following corollary.

Corollary 4.5. *The family $\left\{(-q)^{|\alpha|-\ell(\alpha)}\mathbf{C}_\alpha|_{q\to 1/q}\right\}_\alpha$ exhibits the h-positivity phenomenon.*

5. The expansion of $\mathbf{H}_\alpha$

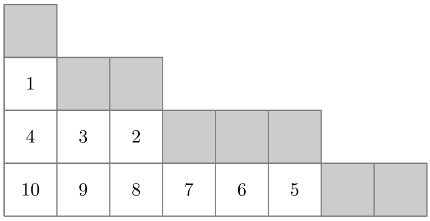
We will now give a combinatorial expansion for $\mathbf{H}_\alpha = \omega\mathbf{B}_\alpha$ in terms of the homogeneous basis. Given a composition $\alpha = (\alpha_1,\ldots,\alpha_\ell) \models n$, let

$$\widetilde{\alpha} = (n, n-\alpha_1, n-\alpha_1-\alpha_2, \ldots, n-\alpha_1-\cdots-\alpha_{\ell-1}),$$

and let

$$\widehat{\alpha} = (n-\alpha_1, n-\alpha_1-\alpha_2, \ldots, n-\alpha_1-\cdots-\alpha_{\ell-1}),$$

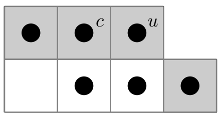
which is contained in $\widetilde{\alpha}$. For instance, we have displayed here $(2,\widetilde{3,2},1)$ (which includes the shaded region) and $(2,\widehat{3,2},1)$ (the region with labels):



In order to interpret $\mathbf{H}_\alpha$ combinatorially, we will first begin by considering $H_{\alpha_{\ell-1}} h_{\alpha_{\ell(\mu)}}$. Using Equation 4, choose some $0 \leq r \leq \alpha_\ell$. Remove r cells from α_ℓ and add these cells to $h_{\alpha_\ell - 1}$. If $r > 0$, the weight associated to such a choice is uq^{r-1}.

We will denote this in pictures by drawing the partition $(\widetilde{\alpha_{\ell-1},\alpha_\ell})$, placing "stones" in $(\widetilde{\alpha_{\ell-1},\alpha_\ell})/\widehat{\alpha}$, and placing r consecutive stones in $(\widehat{\alpha_{\ell-1},\alpha_\ell})$ starting from the right. The first stone placed in $(\widehat{\alpha_{\ell-1},\alpha_\ell})$ (from the right) will be given weight u, and the following $r-1$ stones will be given weight q. These r stones come from the stone directly north, and so we instead label the stones above with u or q.

For instance, to interpret $H_1 h_3$, we can choose $r = 2$ to get uqh_3h_1, described by the following picture:



Note that we placed the labels u and c to depict the stones in the first row which have weights u and q respectively. We use the symbol c to specify that the placed stone is not the first stone placed in its row. We will then say that the stone is "connected" to the stone on its right.

We iterate this process using Equation 4, where at each step, we use the stones to determine the new homogeneous symmetric function which is formed. This process is described by the set of stone placements P_α, which is defined by the following given rules:

A placement of stones in $\widehat{\alpha}$ is a subset of the cells of $\widehat{\alpha}$. Given a placement, and a stone in cell b, we will let $N(b)$, $E(b)$, $S(b)$ be the first stone north, east, or south of b, respectively. (Some of these may not exist for a given stone.)

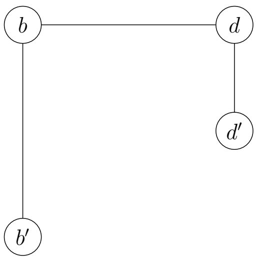
Let P_α be the set of stone placements in $\widetilde{\alpha}$ that satisfy the following conditions:

(1) Every cell of $\widetilde{\alpha}/\widehat{\alpha}$ has a stone.
(2) Let b and $E(b) = d$ be two cells in the same row of $\widetilde{\alpha}$ with a stone. Then $S(b)$ (if it exists) is in a weakly lower row than $S(d)$. If $S(d)$ is in the same

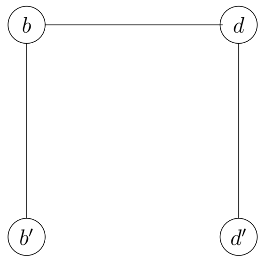
row as $S(b)$, then we will say that b is connected. Otherwise, we will say that b is unconnected. (So b is unconnected if $E(b)$ doesn't exist, as well.)

(3) If a cell b contains a stone and there is no stone placed below (so that $S(b)$ doesn't exist), then we will say that b is a terminal cell.

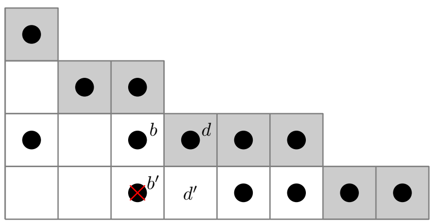
An unconnected stone will get a weight of u and a connected stone will get a weight of q. Diagrammatically, unconnected and connected moves, respectively, can be identified as follows:



and

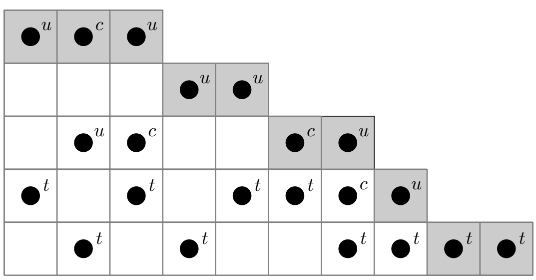


where there are no stones lying along the drawn lines. Here we have attempted to place stones. However, the stone marked with a red cross depicts an illegal stone placement.



The placement is illegal because the stone above, labeled b, has a stone directly on its right, labeled d. Yet no stone is placed below d and in a row weakly above b'. The placement of b' can only happen if there is a stone on d', which is not done. Therefore, the placement is illegal.

Here is perhaps a more interesting example. We have labeled the stones with a u if they correspond to unconnected moves and with a c if they are connected. We will also use a t to label the terminal cells.



For a placement P with a_i terminal cells in row i, we let $\lambda(P)$ be the partition formed by rearranging $a_1, \ldots, a_{\ell(\alpha)}$ in weakly increasing order. Let $\mathrm{con}(P)$ be the number of connected stones in P and $\mathrm{unc}(P)$ the number of unconnected stones in P. Then, as a consequence of Equation 4, we have the following proposition.

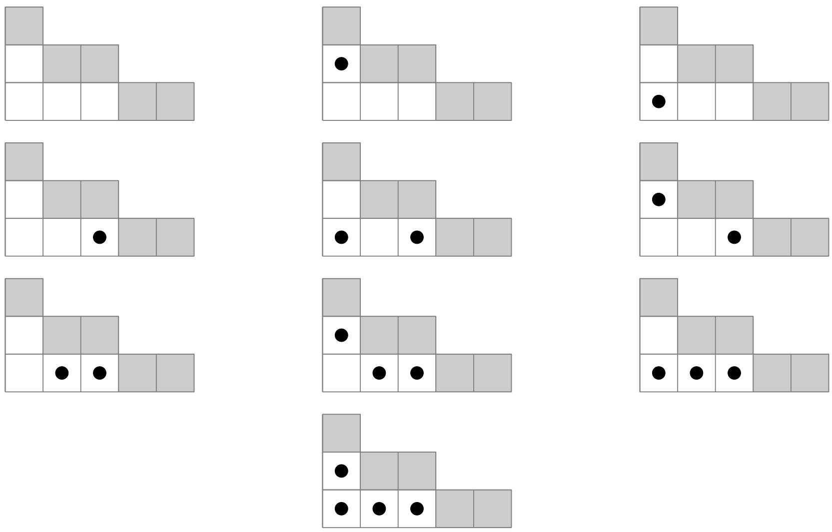
Proposition 5.1.

$$\mathbf{H}_\alpha = \sum_{P \in P_\alpha} q^{\mathrm{con}(P)} u^{\mathrm{unc}(P)} h_{\lambda(P)}.$$

For instance, our above example would give us this term in the summation:

$$u^7 q^4 h_{6,4}.$$

Let us work through a small example. We will compute $\mathbf{H}_{2,2,1}$ by drawing all placements:



To simplify the images, we omitted the stones placed in the shaded region. This gives, respectively,

$$\begin{array}{rrr} h_{2,2,1} & uh_{3,2} & uh_{3,2} \\ uh_{3,1,1} & u^2h_{4,1} & u^2h_{3,2} \\ uqh_{4,1} & u^2qh_{4,1} & u^2qh_5 \\ & u^2q^2h_5 & \end{array}$$

Therefore

$$\mathbf{H}_{2,2,1} = h_{2,2,1} + (u + u + u^2)h_{3,2} + uh_{3,1,1} + (u^2 + uq + u^2q)h_{4,1} + (u^2q + u^2q^2)h_5.$$

Substituting $u = q - 1$ and changing bases we get

$$\mathbf{H}_{2,2,1} = q^4 s_5 + (q^3 + q^2)s_{4,1} + (q^2 + q)s_{3,2} + qs_{3,1,1} + s_{2,2,1}.$$

Theorem 5.2. *For every composition $\alpha \models n$ and subset $S \subseteq \widehat{\alpha}$, there is a partition $\lambda(S) \vdash n$ depending on α that gives*

$$\mathbf{H}_\alpha\big|_{q\to 1+u} = \sum_{S\subseteq\widehat{\alpha}} u^{|S|} h_{\lambda(S)}.$$

Proof. We will start by substituting q to be $1+u$ in our formula for $\mathbf{H}_\alpha$. Define the set of labeled placements LP_α by the following process: Choose a placement $P \in P_\alpha$. We now label P to form $\widehat{P} \in LP_\alpha$ by labeling each unconnected stone with a u; and, for each connected stone, we choose whether to label it with a 1 or a u.

If the resulting labeled placement is $\widehat{P}$, we set $u(\widehat{P})$ to be the number of u's in $\widehat{P}$, and set $\lambda(\widehat{P}) = \lambda(P)$. Then

$$\mathbf{H}_\alpha\big|_{q\to 1+u} = \sum_{\widehat{P}\in LP_\alpha} u^{u(\widehat{P})} h_{\lambda(\widehat{P})}.$$

We will now give a bijection

$$\phi : \mathcal{P}(\widehat{\alpha}) \to LP_\alpha$$

between subsets of $\widehat{\alpha}$ and labeled placements. Let $S \subseteq \widehat{\alpha}$. To construct $\phi(S) = P$, we start with drawing $\widetilde{\alpha}$ and placing a stone in each cell of $\widetilde{\alpha}/\widehat{\alpha}$.

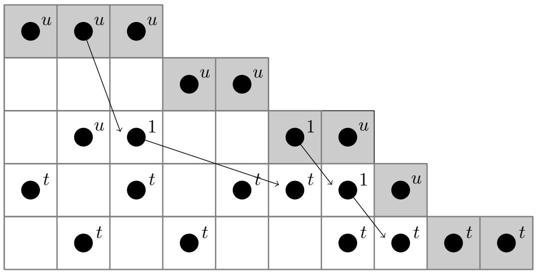
Now start scanning the cells of row $\ell(\alpha)-1$ from left to right. This is the first row of $\widehat{\alpha}$ from the top. Find the first cell in S, if it exists, and place a stone. Now every cell to its right must also have a stone, which we place. For each two adjacent stones we place, say b' and d', there are two adjacent stones in the row above, say b and d. If d' is an element of S, then we mark b with a u. Otherwise, we mark b with a 1. If b has no stone to its right, then we label b with a u since the stone is unconnected. We now move to the next row below.

Suppose we are now scanning the i^{th} row from the top, corresponding to row $\ell(\alpha)-i$ of the partition $\widehat{\alpha}$. Again, for a given cell b, let $N(b)$, $E(b)$, $S(b)$ be the first stone north, east, or south of b, respectively. Let b' be the first cell in S, from the left. Then place a stone here. Now if $SEN(b') = d'$ exists and is in a row higher than b', then we are done for now and we label $b = N(b')$ with a u. Otherwise, we place a stone in the cell d' which is in the same row as b' and the same column as $E(b)$. If d' is in S then we label b with a u, otherwise, we label b with a 1. Now repeat this process with d' instead of b' and look at $SEN(d')$.

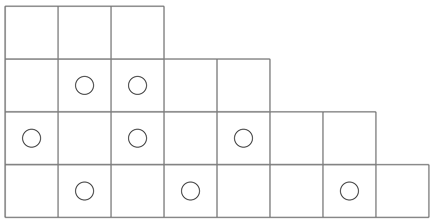
Once this process ends, we may have some more elements of S in this row with no stone. Repeat the process: find the first element b' of S with no stone and place a stone. Look at $SEN(b')$, and so on.

The inverse of ϕ can also be described. Let $P \in LP_\alpha$. Start with S being the stone placement in the $\widehat{\alpha}$ portion of P. For any two cells b' and $E(b') = d'$ that also have $b = N(b')$ in the same row as $d = N(d')$ (in other words, b is connected to d), we remove d' from S if b has a stone labeled with a 1.

Here is an example:



We have drawn an arrow to show which labels decide which stones become elements of S or not. Note that the arrows begin from the cells which were originally labeled with a c to denote that they represent connected stones. If the initial stone is now labeled by 1, then we remove the stone that is at the end of the arrow. If the initial stone is labeled by a u, then we we keep the stone at the end of the arrow. Here we get the following set S:



Note that every u which appears as a label has a corresponding element in S. We then have $|S| = u(P)$, and we set $\lambda(S) = \lambda(\phi(S))$. This particular subset S from our example would give the term

$$u^8 h_{6,4}.$$

This completes the proof. □

6. Some representation theoretical aspects

One of the ways h-positivity arises naturally is by looking at transitive modules of the symmetric group whose stabilizer is a Young subgroup

$$S_\mu = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_{\ell(\mu)}}.$$

Young observed [45] that if M is transitive, meaning $M = S_n v$ is the orbit of a single element $v \in M$, and if the stabilizer of v is isomorphic to S_μ for some $\mu \vdash n$, then

$$M \simeq \operatorname{Ind}_{S_\mu}^{S_n} 1$$

is the induced trivial representation from S_μ to S_n, and

$$\mathcal{F} M = h_\mu.$$

To see an example, let $v = (v_1, \dots, v_n) \in \mathbb{C}^n$ be a vector such that a_i appears μ_i times and $a_1, \dots, a_{\ell(\mu)}$ are all distinct. We will say that v is of type μ. The action of σ permutes the entries of the vectors. Since v is stabilized by a subgroup isomorphic to S_μ, we have that

$$\mathcal{F}\, S_n v = h_\mu.$$

Such a module has a single trivial representation, given by the sum of the elements in the orbit of v. This can be seen from the symmetric function side, by noting that, for any μ, one has

$$\langle h_\mu, s_n \rangle = 1.$$

The representation theoretical setting of Hall–Littlewood polynomials can be found through the work of Steinberg [43], Hotta and Springer [32], Kraft [34], and De Concini and Procesi [12]. These proofs rely on algebraic geometry. De Concini and Procesi give an elementary setting in which to describe the cohomology ring of the variety of flags fixed by a unipotent matrix. This work was further described by Tanisaki [44] in terms of the Tanisaki generators. They can be defined as follows.

For a subset $S \subseteq \{x_1, \dots, x_n\}$, let $e_k(S)$ be the k^{th} elementary symmetric function in the elements of S. Set $d_k(\mu) = \mu'_1 + \cdots + \mu'_k$, where, again, μ' is the conjugate partition of μ. Then the Tanisaki ideal I_μ is generated by the following elements:

$$\left\{ e_r(S) : k \geq r > k - d_k(\mu),\, |S| = k,\, S \subseteq \{x_1, \dots, x_n\} \right\}.$$

Then

$$q^{n(\lambda)} \mathbf{H}_\lambda[X; 1/q] = \mathcal{F}\, \frac{\mathbb{C}[x_1, \dots, x_n]}{I_\mu}.$$

Garsia and Procesi introduce a method to prove this in an accessible manner. In particular, they show that Jing's operators, as presented here, give the Frobenius characteristic of these modules. The orbit harmonics associated to Hall–Littlewood polynomials can be described as follows. Let $v \in \mathbb{C}^n$ be a vector of type λ. Let $W = S_n v$ be the orbit of v, and let $I_W \subset \mathbb{C}[x_1, \dots, x_n]$ be the ideal of polynomials which vanish on W. Then

$$\widetilde{\mathrm{H}}_\lambda[X; 0, q] = q^{n(\lambda)} \mathbf{H}_\lambda[X; 1/q] = \mathcal{F}\, \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathrm{gr}(I_W)},$$

where $\mathrm{gr}(I_W)$ is the ideal generated by the highest homogeneous components of elements in I_W. This method was introduced as an approach for resolving Garsia and Haiman's conjectures regarding the positivity of modified Macdonald polynomials, leading to a development of the theory of orbit harmonics in Garsia and Haiman's research monograph [16]. Through the use of orbit harmonics, one can view the modified Macdonald basis through a similar quotient given by the vanishing ideal of a certain orbit of points. For a better account of this story, we refer the reader to [30].

For general compositions α, $\mathbf{H}_\alpha$ is not Schur positive when expanded as a polynomial in q. For instance,

$$\mathbf{H}_{(1,3)} = q^3 s_4 + q^2 s_{3,1} + (q-1) s_{2,2}.$$

However, under the substitution $q \to 1+u$, we get an h-positive expression, and therefore a Schur positive polynomial. Given a composition $\alpha \models n$, let M_α be a module whose Frobenius characteristic, as a polynomial in u, gives

$$\mathcal{F}\, M_\alpha = \mathbf{H}_\alpha \big|_{q \to 1+u}.$$

It is then interesting to ask for a natural family of modules, indexed by compositions, which gives this Frobenius characteristic.

Proposition 6.1. *The number of orbits in M_α is $2^{n(\alpha)}$, where*

$$n(\alpha) = \alpha_2 + 2\alpha_3 + 3\alpha_4 + \cdots + (\ell(\alpha)-1)\alpha_\ell = \sum_{i=1}^{\ell(\alpha)} (i-1)\alpha_i$$

is the size of $\widehat{\alpha}$. In particular, the Hilbert series of invariants is the polynomial $(1+u)^{n(\alpha)}$.

Proof. Given that $\mathcal{F}\, M_\alpha = \mathbf{H}_\alpha \big|_{q\to 1+u}$, we can find the number of orbits by taking the homogeneous expansion of $\mathbf{H}_\alpha \big|_{q \to 1+u}$ and replacing each h_μ by 1. This gives that the Hilbert series for the invariants is given by

$$\sum_{S \subseteq \widehat{\alpha}} u^{|S|} = (1+u)^{|\widehat{\alpha}|}. \qquad \square$$

Note that when α is a partition λ, we have $\langle \mathbf{H}_\lambda, h_n \rangle = q^{n(\lambda)}$, which is one of the defining properties of Hall–Littlewood polynomials. But in fact, it holds in general that, for any composition, one has

$$\langle \mathbf{H}_\alpha, h_n \rangle = q^{n(\alpha)}.$$

7. An expansion for the Delta Theorem when $t = 0$

The Delta Theorem, conjectured in [25] and proved by D'Adderio and Mellit [11], gives a monomial expansion of the symmetric function $\Delta_{e_k} e_n$ or $\Delta'_{e_k} e_n$, where, for any symmetric function F, one defines Δ_F and Δ'_F as eigenoperators of the modified Macdonald basis by setting

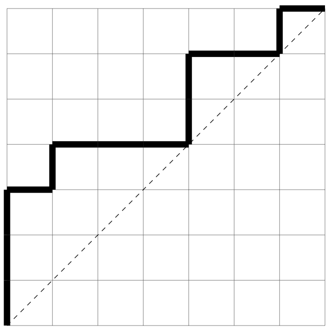
$$\Delta_F = F[B_\mu]\, \widetilde{\mathrm{H}}_\mu \quad \text{and} \quad \Delta'_F = F[B_\mu - 1]\, \widetilde{\mathrm{H}}_\mu, \qquad \text{with } B_\mu = \sum_{(r,s)\in\mu} q^r t^s.$$

Delta operators were first defined in [5] in order to study the operator ∇. In particular, one has, for $\mu \vdash n$,

$$\Delta'_{e_{n-1}} \widetilde{\mathrm{H}}_\mu = \Delta_{e_n} \widetilde{\mathrm{H}}_\mu = e_n[B_\mu]\, \widetilde{\mathrm{H}}_\mu = T_\mu\, \widetilde{\mathrm{H}}_\mu = \nabla\, \widetilde{\mathrm{H}}_\mu\,.$$

We should also note that the Extended Delta Theorem was also proved in [6], and it gives a combinatorial interpretation for $\Delta_{h_l} \Delta'_{e_k} e_n$.

The Delta Theorem can be stated in terms of Column LLT polynomials. Named after Lascoux, Leclerc and Thibon [35], LLT polynomials now play an important role in the theory of Macdonald polynomials. Amongst their important properties, they are known to be Schur positive [21]. We now introduce the LLT polynomial corresponding to a Dyck path. Let D be a Dyck path in the $n \times n$ square, given by a sequence of north and east unit steps which stay weakly above the main diagonal. For instance,



gives a Dyck path in D_7. For $D \in D_n$, we let $a(D) = (a_1(D), \ldots, a_n(D))$ be the area sequence, where $a_i(D)$ is the number of unit lattice cells between the i^{th} north step of the path and the main diagonal. The above example would have area sequence $(0, 1, 2, 2, 0, 1, 0)$. The area of the Dyck path, $\operatorname{area}(D)$, is the sum of the area numbers. A labeled Dyck path $L \in LD_n$ is a Dyck path whose north steps are labeled with positive integers so that the columns of L are increasing from bottom to top. The area of L is the area of the supporting Dyck path. If the labels of L are given by $l_1, \ldots, l_n$ as we read from bottom to top, then the number of diagonal inversions is given by

$$\operatorname{dinv}(L) = |\{\ i < j : l_i < l_j \text{ and } a_i = a_j, \text{ or } l_i > l_j \text{ and } a_i = a_j + 1\}|.$$

Denote by x_L the monomial we get from all the labels: $x_L = x_{l_1} \cdots x_{l_n}$. The Column LLT polynomial associated to a Dyck path $D \in D_n$ is given by

$$LLT_D = \sum_{L \,\in\, LD_n \text{ of shape } D} q^{\operatorname{dinv}(L)} x_L.$$

These polynomials were proved to exhibit the e-positivity phenomenon [9], and they have a combinatorial description given by Alexandersson and Sulzgruber [2].

The final part we need in order to describe the Delta Theorem is Haglund's factor, which is given by

$$H_D(z) = \prod_{a_{i+1}(D) = a_i(D)+1} \left(1 + \frac{z}{t^{a_i(D)}}\right).$$

The rise version of the Delta Theorem can now be stated.

Theorem 7.1 (D'Adderio and Mellit [11])**.**

$$\sum_{D\in LD_n} t^{\mathrm{area}(D)} q^{\mathrm{dinv}(D)} H_D(z) x_D = \sum_{k=0}^{n-1} z^k \Delta'_{e_{n-1-k}} e_n$$

Zabrocki conjectures in [46] that this symmetric function is the Frobenius characteristic of the following natural S_n-module. Let $\theta_1, \ldots, \theta_n$ be a set of anti-commuting variables that commute with $x_1, \ldots, x_n, y_1, \ldots, y_n$. Let $R_n^{(1)}$ be the quotient of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n, \theta_1, \ldots, \theta_n]$ by the ideal generated by S_n invariants with zero constant term. Then the conjecture is that

$$\mathcal{F} R_n^{(1)} = \sum_{k=0}^{n-1} z^k \Delta'_{e_{n-1-k}} e_n,$$

where z gives the grading for the anti-commuting set of variables.

The $t = 0$ version of the Delta Theorem was proved in [14]. On the representation theoretical side, Haglund, Rhoades and Shimozono [26] showed that

$$(\mathrm{rev}_q\, \omega) \Delta'_{e_{k-1}} e_n \Big|_{t=0} = \mathcal{F} \frac{\mathbb{C}[x_1, \ldots, x_n]}{(x_1^k, \ldots, x_n^k, e_n, \ldots, e_{n-k+1})},$$

where rev_q reverses the powers of q appearing in the expression. We should also mention that Pawlowski and Rhoades produced a flag variety whose cohomology ring also gives this Frobenius characteristic [41]. The following observation shows that the Frobenius characteristic of these modules exhibits the h-positivity phenomenon.

It was noted in [13] that this symmetric function has a nice expansion in terms of B operators, namely

$$\Delta'_{e_{k-1}} e_n \Big|_{t=0} = \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} \mathbf{B}_\alpha.$$

It follows from the work here that we have the next theorem.

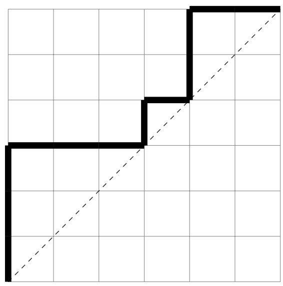
Theorem 7.2.

$$\Delta'_{e_{k-1}} e_n \Big|_{t=0} \Big|_{q\to 1+u} = \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} \mathbf{B}_\alpha = \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} \sum_{S\subseteq \widehat{\alpha}} u^{|S|} e_{\lambda(S)}.$$

One of the beautiful facts that follow from this result is the connection between the sum of LLT polynomials for balanced paths and the sum of Hall–Littlewood polynomials corresponding to compositions of n. This was first seen in [13], but is made more explicit by our combinatorial formula for $\mathbf{B}_\alpha$. Note that from Haglund's factor, one has that, for any Dyck path D,

$$t^{\mathrm{area}(D)} H_D(z) \Big|_{z^{n-k}} \Big|_{t=0} = t^{\mathrm{area}(D)} \prod_{a_{i+1}(D)=a_i(D)+1} \left(1 + \frac{z}{t^{a_i(D)}}\right) \Big|_{z^{n-k}} \Big|_{t=0} = 0$$

whenever there is a north step in D that is first in its column but does not begin on the main diagonal, or the path has more than k north segments. This means we get a nonzero evaluation only when D is a balanced path which returns to the diagonal k times. A balanced path is a Dyck path whose north segments all begin on the main diagonal. The north and east steps of such a path are given by $N^{\alpha_1}E^{\alpha_1}\cdots N^{\alpha_k}E^{\alpha_k}$ for some composition $\alpha = (\alpha_1, \dots, \alpha_k) \models n$. Denote this balanced path by BP_α. For instance,



is the balanced path corresponding to the composition $(3, 1, 2)$. Then one obtains the following corollary.

Corollary 7.3.

$$\sum_{\substack{\alpha\models n\\ \ell(\alpha)=k}} LLT_{BP_\alpha} = \sum_{\substack{\alpha\models n\\ \ell(\alpha)=k}} \sum_{S\subseteq\widehat{\alpha}} u^{|S|} e_{\lambda(S)}.$$

The LLT polynomial of a balanced path is closely connected to Hall–Littlewood polynomials. For this connection, we refer the reader to Haglund's book [22], which also contains further important results and combinatorial descriptions related to this subject.

Relating back to delta operators, Haglund, Rhoades, and Shimozono also observed the following.

Theorem 7.4 (Haglund, Rhoades, and Shimozono [27])**.**

$$\omega\Delta'_{s_\nu} e_n\big|_{t=0} = \sum_{k=\ell(\nu)+1}^{|\nu|+1} P_{\nu,k-1}(q) \sum_{\substack{\mu\vdash n\\ \ell(\mu)=k}} q^{n(\mu)-\binom{\ell(\mu)}{2}} \begin{bmatrix} k \\ m_1(\mu), \dots, m_n(\mu)\end{bmatrix}_q \mathbf{H}_\mu,$$

with

$$P_{\nu,k-1}(q) = q^{|\nu|-\binom{k}{2}} \sum_{\substack{|\rho|=|\nu|\\ \ell(\rho)=k-1}} q^{n(\rho)} \begin{bmatrix} k-1 \\ m(\rho)\end{bmatrix}_q K_{\nu,\rho}(q),$$

where $K_{\nu,\rho}$ is the Kostka–Foulkes polynomial appearing in the expansion

$$\mathbf{H}_\rho = \sum_\nu K_{\nu,\rho}(q) s_\nu.$$

They also give a representation theoretical setting for these symmetric functions. Since all the above expressions are polynomials in q, with positive integer coefficients, we get the following theorem.

Theorem 7.5. *The family* $\left\{\Delta'_{s_\nu} e_n\big|_{t=0}\right\}$ *exhibits the e-positivity phenomenon.*

As conjectured in [42], it still remains open to understand the following more general phenomenon.

Conjecture 7.6. *The family of symmetric functions* $\{\Delta_{s_\lambda} e_n\}_\lambda$ *exhibits the e-positivity phenomenon.*

In this case, there are two variables, q and t, and substituting either by $1+u$ would conjecturally give an e-positive expression.

8. Final remarks

An interesting question arises from these observations. Given a graded module M, when would we know that $\mathcal{F}M$ exhibits the h-positivity (or e-positivity) phenomenon? Many of the symmetric functions we have seen appear as quotients of polynomial rings. It is not clear why these quotients would exhibit this phenomenon, and it would be worthwhile to investigate why this is the case. From a representation theoretical point of view, the modules associated to the homogeneous or elementary basis are simpler to understand. From the symmetric function side, it is easier to get a Schur expansion from homogeneous (or elementary) basis expansions by using Kostka coefficients, which follow from the Pieri rules. The usual expansions which appear in the literature are in terms of monomial symmetric functions. To go from the monomial basis to the Schur basis would involve inverting the Kostka coefficients, which is not as simple to understand.

The positivities presented here are only a glimpse of the positivity phenomenon which seems to exist at a larger scale, specifically when it comes to symmetric functions appearing in the theory of modified Macdonald polynomials. Understanding why these positivities hold may give better insight into analyzing these symmetric functions, possibly leading to an understanding of their Schur basis expansions and therefore the decomposition of their corresponding modules.

Acknowledgments

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