REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 64, No. 2, 2023, Pages 333–353
Published online: March 13, 2023
https://doi.org/10.33044/revuma.2807

HOMOGENEOUS WEIGHT ENUMERATORS OVER INTEGER RESIDUE RINGS AND FAILURES OF THE MACWILLIAMS IDENTITIES

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In memoriam: Tadashi Nagano, 1930–2017

Abstract. The MacWilliams identities for the homogeneous weight enumerator over $\mathbb{Z}/m\mathbb{Z}$ do not hold for composite $m \geq 6$. For such m, there exist two linear codes over $\mathbb{Z}/m\mathbb{Z}$ that have the same homogeneous weight enumerator, yet whose dual codes have different homogeneous weight enumerators.

1. Introduction

Suppose R is a finite ring with 1, and let w be an integer-valued weight on R. That is, w is a function $w : R \to \mathbb{Z}$, with $w(0) = 0$ and $w(r) > 0$ for $r \neq 0$. Write $w_{\max}$ for the largest value of w on R. The weight w extends additively to a weight on R^n, $w : R^n \to \mathbb{Z}$, by $w(r_1, r_2, \ldots, r_n) = \sum_{i=1}^n w(r_i)$.

A left *linear code* over R of length n is a left R-submodule $C \subseteq R^n$. The weight w determines the *w-weight enumerator* of a linear code C:

$$\operatorname{wwe}_C(X, Y) = \sum_{c \in C} X^{nw_{\max} - w(c)} Y^{w(c)} = \sum_{j=0}^{nw_{\max}} A_j(C) X^{nw_{\max} - j} Y^j,$$

where $A_j(C)$ is the number of codewords $c \in C$ having $w(c) = j$. (We will write $A_j^w(C)$ if the weight w is not obvious from context.) To save space later in the paper, we will sometimes write wwe_C with $X = 1$ and $Y = t$, so that

$$\operatorname{wwe}_C = \sum_{c \in C} t^{w(c)} = \sum_{j=0}^{nw_{\max}} A_j(C) t^j.$$

In R^n, define the *standard dot product* by $x \cdot y = \sum_{i=1}^n x_i y_i \in R$ for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R^n$. Given a linear code $C \subseteq R^n$, define its (right) *dual code* by

$$C^\perp = \{y \in R^n : x \cdot y = 0 \text{ for all } x \in C\}.$$

In favorable circumstances one can express the weight enumerator of $C^\perp$ in terms of the weight enumerator of C, via a linear change of variables. The famous

2020 *Mathematics Subject Classification.* 94B05.
Key words and phrases. Homogeneous weight, MacWilliams identities, dual code, counter-examples.

'MacWilliams identities' do exactly that when w is the Hamming weight and R is a finite field [10, 11] or a finite Frobenius ring [12].

This paper discusses the situation for the homogeneous weight on $\mathbb{Z}/m\mathbb{Z}$. The main result is the failure of the MacWilliams identities for the homogeneous weight enumerator (howe) over $R = \mathbb{Z}/m\mathbb{Z}$ for composite $m \geq 6$. More specifically, we prove the existence of two linear codes C, D over $\mathbb{Z}/m\mathbb{Z}$ such that $\mathrm{howe}_C = \mathrm{howe}_D$, yet $\mathrm{howe}_{C^\perp} \neq \mathrm{howe}_{D^\perp}$, because $A_j(C^\perp) \neq A_j(D^\perp)$ for some j.

2. Homogeneous weight in general

The homogeneous weight for $\mathbb{Z}/m\mathbb{Z}$ was first defined by Constantinescu and Heise [3]; the definition was generalized to all finite rings in [6] and [8].

Let R be a finite ring with 1. We present a formula for the homogeneous weight, as found in [6]. Let $\mathcal{P}$ be the poset of principal left ideals of R under set containment, and let μ be the Möbius function of $\mathcal{P}$. Denote the group of units of R by $\mathcal{U} = \mathcal{U}(R)$. For $r, s \in R$, it is known that $Rs = Rs$ if and only if $\mathcal{U}r = \mathcal{U}s$, [12, Proposition 5.1]. For a positive real number ζ, the homogeneous weight of average weight ζ on R is given by

$$\mathrm{w}(x) = \zeta\left(1 - \frac{\mu(0, Rx)}{|\mathcal{U}x|}\right). \tag{2.1}$$

For a fixed $\zeta > 0$, the homogeneous weight w is characterized by three properties:

(1) $\mathrm{w}(0) = 0$;
(2) w is constant on left $\mathcal{U}$-orbits: $\mathrm{w}(ux) = \mathrm{w}(x)$ for all $u \in \mathcal{U}$ and $x \in R$;
(3) the average weight on nonzero principal left ideals is the constant ζ:

$$\sum_{y \in Rx} \mathrm{w}(y) = \zeta|Rx|,$$

for any $Rx \neq 0$.

In addition, if R is Frobenius, then the average weight on *any* nonzero left ideal is ζ (and conversely) [6, Corollary 1.6].

It is typical to set $\zeta = 1$, and the resulting values of w are rational numbers, but not usually integers. In order to ensure that w has integer values, we will choose ζ so as to clear the denominators in (2.1).

The homogeneous weight of a finite field $\mathbb{F}_q$, with $\zeta = (q-1)/q$, equals the Hamming weight. As the MacWilliams identities hold for the Hamming weight over finite fields, they also hold for the homogeneous weight enumerator over finite fields—in particular, over the prime fields $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$, p prime.

The homogeneous weight of $\mathbb{Z}/4\mathbb{Z}$, with $\zeta = 1$, equals the Lee weight, so the MacWilliams identities hold also in that context [7, Equation (9)].

3. Prime powers

In this section we consider $R = \mathbb{Z}/p^a\mathbb{Z}$, with p prime. If $a = 1$, then $\mathbb{Z}/pZ$ is a finite field. As we saw at the end of the previous section, the homogeneous weight on a finite field is just a multiple of the Hamming weight, and the MacWilliams identities hold. Similarly, when $p = a = 2$, the homogeneous weight on $\mathbb{Z}/4\mathbb{Z}$ is the Lee weight, and again the MacWilliams identities hold. For all other cases, we will see that the MacWilliams identities fail for the homogeneous weight: there exist two linear codes whose homogeneous weight enumerators are equal, yet their dual codes have different homogeneous weight enumerators.

The ideals of $\mathbb{Z}/p^a\mathbb{Z}$ form a chain:

$$\mathbb{Z}/p^a\mathbb{Z} = (1) = (p^0) \supset (p) \supset (p^2) \supset \cdots \supset (p^{a-1}) \supset (p^a) = (0).$$

The ideals have size

$$|(p^i)| = p^{a-i}. \tag{3.1}$$

Choosing $\zeta = p - 1$, one obtains the following values for w:

$$\mathrm{w}(a) = \begin{cases} 0, & a = 0, \\ p, & a \in (p^{a-1}) - \{0\}, \\ p-1, & a \notin (p^{a-1}). \end{cases}$$

Remark 3.1. Call a vector with exactly one nonzero entry a *singleton*; a vector with exactly two nonzero entries is a *doubleton*. Over $R = \mathbb{Z}/p^a\mathbb{Z}$, note that any vector $v \in R^n$ with $\mathrm{w}(v) = p - 1$ must be a singleton with nonzero entry $r \notin (p^{a-1})$. Similarly, any vector v with $\mathrm{w}(v) = p$ must be a singleton with nonzero entry $r \in (p^{a-1}) - (0)$, or (only when $p = 2$) v is a doubleton with both nonzero entries not in (p^{a-1}). I call the $p = 2$ exception the 'curse of small values.'

For $a \geq 2$, define two linear codes over $\mathbb{Z}/p^a\mathbb{Z}$ via the following two generator matrices of sizes $1 \times (p+1)$ and $2 \times (p+1)$, respectively:

$$\begin{aligned} G_1 &= \begin{bmatrix} p^{a-1} & p^{a-2} & p^{a-2} & p^{a-2} & \cdots & p^{a-2} \end{bmatrix}, \\ G_2 &= \begin{bmatrix} 0 & p^{a-1} & p^{a-1} & p^{a-1} & \cdots & p^{a-1} \\ p^{a-1} & 0 & p^{a-1} & 2p^{a-1} & \cdots & (p-1)p^{a-1} \end{bmatrix}. \end{aligned} \tag{3.2}$$

Theorem 3.2. *Suppose p is a prime and $a \geq 2$ is an integer. If C_1 and C_2 are the linear codes over $\mathbb{Z}/p^a\mathbb{Z}$ generated by* (3.2), *then* $\mathrm{howe}_{C_1} = \mathrm{howe}_{C_2} = 1+(p^2-1)t^{p^2}$. *If $a = p = 2$, then* $\mathrm{howe}_{C_1^\perp} = \mathrm{howe}_{C_2^\perp}$. *In all other cases,* $\mathrm{howe}_{C_1^\perp} \neq \mathrm{howe}_{C_2^\perp}$. *The smallest weights at which the numbers of dual codewords differ appear in the table below.*

Code	p	a	A_{p-1}	A_p
$C_1^\perp$	≥ 3	2		$p-1$
$C_2^\perp$				p^2-1
$C_1^\perp$	≥ 2	≥ 3	$2p^{a-1} - p^2 - p$	
$C_2^\perp$			$p^a + p^{a-1} - p^2 - p$	

Proof. The codewords of C_1 have the form rG_1 for $r \in \mathbb{Z}/p^2\mathbb{Z}$, because p^2 annihilates G_1; $|C_1| = p^2$. If r is a unit, then $\mathrm{w}(rG_1) = \mathrm{w}(p^{a-1}) + p\mathrm{w}(p^{a-2}) = p + p(p-1) = p^2$. There are $p^2 - p$ units. If r has the form $r = up$, u a unit, then $\mathrm{w}(upG_1) = p\mathrm{w}(p^{a-1}) = p^2$. There are $p-1$ elements of the form up. Thus $\mathrm{howe}_{C_1} = 1 + (p^2-1)t^{p^2}$.

The codewords of C_2 have the form $[r\ s]G_2$ with $r, s \in \mathbb{Z}/p\mathbb{Z}$; $|C_2| = p^2$. For $r \neq 0$ (a unit) and $s = 0$, $\mathrm{w}([r\ 0]G_2) = p\mathrm{w}(p^{a-1}) = p^2$. There are $p-1$ such codewords. For arbitrary r and $s \neq 0$, there are exactly p entries of $[r\ s]G_2$ that are unit multiples of p^{a-1} and one entry equal to zero (in the position indexed by $-rs^{-1}$). Then $\mathrm{w}([r\ s]G_2) = p^2$, and there are $p(p-1) = p^2 - p$ codewords of this form. Thus $\mathrm{howe}_{C_2} = 1 + (p^2-1)t^{p^2}$.

When $a = p = 2$, [7, Equation (9)] implies that $\mathrm{howe}_{C_1^\perp} = \mathrm{howe}_{C_2^\perp}$.

Continue with $a = 2$, but allow any $p \geq 2$. All $a \in \mathbb{Z}/p^2\mathbb{Z}$ with $\mathrm{w}(a) = p-1$ are units. Any vector v with $\mathrm{w}(v) = p-1$ is a singleton with nonzero entry being a unit; see Remark 3.1. There being no zero-columns in the two generator matrices, we conclude that $A_{p-1}(C_1^\perp) = A_{p-1}(C_2^\perp) = 0$. Now consider $A_p(C_1^\perp)$ and $A_p(C_2^\perp)$. If $p > 2$, any vector of weight p is a singleton with nonzero entry from $(p) - (0)$. (When $p = 2$, see Remark 3.3.) Such a singleton will annihilate G_1 if and only if its nonzero entry is in the first position. This implies that $A_p(C_1^\perp) = |(p)-(0)| = p-1$. On the other hand, a singleton with a nonzero entry from $(p)-(0)$ (in any position) always annihilates C_2, so that $A_p(C_2^\perp) = (p-1)(p+1) = p^2-1$.

Now assume $a \geq 3$. A vector v with $\mathrm{w}(v) = p-1$ must be a singleton with nonzero entry $r \in R - (p^{a-1})$. Such a singleton annihilates G_1 when r annihilates the entry of G_1 in its position. This happens when $r \in (p) - (p^{a-1})$ is in the first position or when $r \in (p^2) - (p^{a-1})$ is in any of the last p positions. Using (3.1), we see that $A_{p-1}(C_1^\perp) = (p^{a-1} - p) + p(p^{a-2} - p)$, which simplifies as claimed.

Similarly, a singleton v with nonzero entry $r \in R-(p^{a-1})$ will annihilate G_2 when r annihilates the column of G_2 in its position. This happens when $r \in (p) - (p^{a-1})$ is in any of the $p+1$ possible positions. Thus $A_{p-1}(C_2^\perp) = (p+1)(p^{a-1} - p)$, as claimed. We used $a \geq 3$ to ensure that $(p) - (p^{a-1})$ is nonempty. □

Remark 3.3. Consider the case $p = 2$, $a = 2$ more closely. When $p = 2$, there are doubletons of weight 2 with exactly two unit entries (the 'curse of small values' from Remark 3.1). Doubletons of this form account for an additional 2 elements of $A_2(C_1^\perp)$. This makes the final count $A_2(C_1^\perp) = 3$, which equals $A_2(C_2^\perp)$.

Remark 3.4. The results of this section also hold for finite commutative chain rings.

Example 3.5. For $m = 8 = 2^3$, $p = 2$, $a = 3$, the generator matrices of (3.2) are

$$G_3 = \begin{bmatrix} 4 & 2 & 2 \end{bmatrix}, \quad G_4 = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \end{bmatrix}.$$

Then $\mathrm{howe}_{C_3} = \mathrm{howe}_{C_4} = 1 + 3t^4$, while a computation gives

$$\mathrm{howe}_{C_3^\perp} = 1 + 2t + 31t^2 + 60t^3 + 31t^4 + 2t^5 + t^6,$$
$$\mathrm{howe}_{C_4^\perp} = 1 + 6t + 15t^2 + 84t^3 + 15t^4 + 6t^5 + t^6.$$

The counts for $A_1(C_*^\perp) = A_{p-1}(C_*^\perp)$ are consistent with Theorem 3.2.

Example 3.6. For $m = 8 = 2^3$, $p = 2$, $a = 3$, there is an earlier example in the literature due to the referee of [14, Example 5.6], that also appears in [1, Example 5.3]. Set

$$G_5 = \begin{bmatrix} 1 & 1 & 4 \end{bmatrix}, \quad G_6 = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

Let C_5 and C_6 be the linear codes over $\mathbb{Z}/8\mathbb{Z}$ generated by G_5 and G_6, respectively. Then $\operatorname{howe}_{C_5} = \operatorname{howe}_{C_6} = 1 + 2t^2 + 5t^4$, but

$$\begin{aligned} \operatorname{howe}_{C_5^\perp} &= 1 + 2t + 7t^2 + 44t^3 + 7t^4 + 2t^5 + t^6, \\ \operatorname{howe}_{C_6^\perp} &= 1 + 2t + 23t^2 + 12t^3 + 23t^4 + 2t^5 + t^6. \end{aligned}$$

Example 3.7. For $m = 9 = 3^2$, $p = 3$, $a = 2$, the generator matrices of (3.2) are

$$G_7 = \begin{bmatrix} 3 & 1 & 1 & 1 \end{bmatrix}, \quad G_8 = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 3 & 0 & 3 & 6 \end{bmatrix}.$$

Then $\operatorname{howe}_{C_7} = \operatorname{howe}_{C_8} = 1 + 8t^9$, while a computation gives

$$\begin{aligned} \operatorname{howe}_{C_7^\perp} &= 1 + 2t^3 + 18t^4 + 18t^5 + 132t^6 + 72t^7 \\ &\quad + 126t^8 + 266t^9 + 72t^{10} + 18t^{11} + 4t^{12}, \\ \operatorname{howe}_{C_8^\perp} &= 1 + 8t^3 + 240t^6 + 464t^9 + 16t^{12}. \end{aligned}$$

The counts for $A_3(C_*^\perp) = A_p(C_*^\perp)$ are consistent with Theorem 3.2.

Example 3.8. For $m = 27 = 3^3$, $p = 3$, $a = 3$, the generator matrices of (3.2) are

$$G_9 = \begin{bmatrix} 9 & 3 & 3 & 3 \end{bmatrix}, \quad G_{10} = \begin{bmatrix} 0 & 9 & 9 & 9 \\ 9 & 0 & 9 & 18 \end{bmatrix}.$$

Then $\operatorname{howe}_{C_9} = \operatorname{howe}_{C_{10}} = 1 + 8t^9$, while a computation gives

$$\begin{aligned} \operatorname{howe}_{C_9^\perp} &= 1 + 6t^2 + 8t^3 + 378t^4 + 36t^5 + 6234t^6 + 1512t^7 \\ &\quad + 36846t^8 + 12452t^9 + 1512t^{10} + 48t^{11} + 16t^{12}, \\ \operatorname{howe}_{C_{10}^\perp} &= 1 + 24t^2 + 8t^3 + 216t^4 + 144t^5 + 6720t^6 + 864t^7 \\ &\quad + 36576t^8 + 13424t^9 + 864t^{10} + 192t^{11} + 16t^{12}. \end{aligned}$$

The counts for $A_2(C_*^\perp) = A_{p-1}(C_*^\perp)$ are consistent with Theorem 3.2.

4. Homogeneous weight on $\mathbb{Z}/m\mathbb{Z}$

As preparation for the situations where m is not a prime power, we look at the homogeneous weight on $\mathbb{Z}/m\mathbb{Z}$ in more detail.

Using formula (2.1), one can develop an explicit formula for w on $R = \mathbb{Z}/m\mathbb{Z}$. If m is not clear from context, we write w_m for the homogeneous weight on $\mathbb{Z}/m\mathbb{Z}$. Denote the prime factorization of m by $m = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1 < p_2 < \cdots < p_k$ are distinct primes and the exponents a_i are positive integers. The ideals of R are generated by divisors of m. Every element of R has the form $x = up_1^{b_1} \cdots p_k^{b_k}$ for some unit $u \in \mathcal{U}$ and integers b_i satisfying $0 \leq b_i \leq a_i$; the b_i are uniquely determined by x. The *socle* of R, denoted by $\operatorname{soc}(R)$, is defined to be the ideal generated by the minimal ideals of R; for $R = \mathbb{Z}/m\mathbb{Z}$, $\operatorname{soc}(R)$ is the ideal generated by $p_1^{a_1-1} \cdots p_k^{a_k-1} = m/(p_1 \cdots p_k)$. Every element of the socle is a unit multiple of some m/f, where f is a square-free polynomial expression in the p_i: $f = p_1^{\epsilon_1} \cdots p_k^{\epsilon_k}$, where each ϵ_i equals 0 or 1. For any such f, define $\delta_f = \{i : \epsilon_i = 1\}$, so that $f = \prod_{i \in \delta_f} p_i$. (The empty product is equal to 1.) Write $\bar{\delta}_f$ for the set complement $\bar{\delta}_f = \{1, 2, \ldots, k\} - \delta_f$.

The next result appeared in [8, Proposition 7]; also see [4, Section 3]. The number $\zeta = \prod_{i=1}^k (p_i - 1)$ may be written ζ_m when m is not clear from context.

Proposition 4.1. *Let $R = \mathbb{Z}/m\mathbb{Z}$, with $m = p_1^{a_1} \cdots p_k^{a_k}$. Then the homogeneous weight* w *with $\zeta = \prod_{i=1}^k (p_i - 1)$ has the form*

$$\mathrm{w}(x) = \begin{cases} \zeta - (-1)^{|\delta_f|} \prod_{j \in \bar{\delta}_f} (p_j - 1), & x = u(m/f) \in \operatorname{soc}(R), \\ \zeta, & x \notin \operatorname{soc}(R). \end{cases} \tag{4.1}$$

In particular, the values of w *are integers.*

Remark 4.2. When $a_1 = a_2 = \cdots = a_k = 1$, $\operatorname{soc}(R) = R$, and there are no elements in R with $\mathrm{w} = \zeta$. If some $a_i > 1$, then there are elements $x \in R$ with $\mathrm{w}(x) = \zeta$. For example, $\mathrm{w}(p_i) = \zeta$ if $a_i > 1$.

The group $\mathcal{U}_m$ of units acts on $\mathbb{Z}/m\mathbb{Z}$ by multiplication. Denote the orbit of $x \in \mathbb{Z}/m\mathbb{Z}$ under this action by $\operatorname{orb}(x)$. Another way to say that $x = u(m/f)$ is that $x \in \operatorname{orb}(m/f)$.

If we let $A = p_1^{a_1-1} \cdots p_k^{a_k-1} = m/(p_1 p_2 \cdots p_k)$, which is the generator of the socle, then any $x = u(m/f) \in \operatorname{soc}(R)$ has the form $x = uA \prod_{j \in \bar{\delta}_f} p_j$.

Proposition 4.3. *Let $R = \mathbb{Z}/m\mathbb{Z}$, with $m = p_1^{a_1} \cdots p_k^{a_k}$, and let* w *be the homogeneous weight with $\zeta = \prod_{i=1}^k (p_i - 1)$. Assume $k \geq 2$. For every $1 \leq i < j \leq k$, $\mathrm{w}(m/(p_i p_j)) < \zeta$. The smallest nonzero value of* w *occurs at unit multiples of $x_0 = m/(p_1 p_2)$. Moreover, $2\mathrm{w}(x_0) \geq \zeta$, with equality occurring if and only if $p_1 = 2$ and $p_2 = 3$.*

Proof. For elements $x = u(m/f) \in \operatorname{soc}(R)$, we see from (4.1) that $\mathrm{w}(x) < \zeta$ when $|\delta_f|$ is even. Similarly, $\mathrm{w}(x)$ is as small as possible when $(-1)^{|\delta_f|} \prod_{j \in \bar{\delta}_f} (p_j - 1)$ is positive and as large as possible. This forces $|\delta_f|$ to be even, the product $\prod_{j \in \bar{\delta}_f} (p_j - 1)$ to have as many terms as possible, and for those terms to be as large as possible. If δ_f is empty, then $\mathrm{w}(x) = 0$. Thus the smallest nonzero value of w occurs when $|\delta_f| = 2$ and $\delta_f = \{1, 2\}$ (since the primes are arranged in ascending order).

To address the inequality $2\mathrm{w}(x_0) \geq \zeta$, note that $2\mathrm{w}(x_0) - \zeta$ simplifies to

$$2\mathrm{w}(x_0) - \zeta = \left(\prod_{j=3}^k (p_j - 1) \right) \left((p_1 - 1)(p_2 - 1) - 2 \right).$$

Since $p_1 \geq 2$ and $p_2 \geq 3$, the result follows. □

Remark 4.4. This remark extends Remark 3.1. Over $R = \mathbb{Z}/m\mathbb{Z}$, m not prime, any doubleton v has $\mathrm{w}(v) \geq 2\mathrm{w}(x_0) \geq \zeta$. Thus, any vector $v \in R^n$ with $\mathrm{w}(v) < \zeta$

must be a singleton. Similarly, any vector v with $\mathrm{w}(v) = \zeta$ must be a singleton or (only when $6 \mid m$) a doubleton with both nonzero entries being unit multiples of x_0. This is another example of the 'curse of small values.'

Given positive integers m and m', we next compare the homogeneous weights of $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/mm'\mathbb{Z}$. Denote by Δ the set of primes that divide m' but do not divide m. Define $\nu_{m'} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/mm'\mathbb{Z}$ by $\nu_{m'}(x) = m'x$ for $x \in \mathbb{Z}/m\mathbb{Z}$.

Lemma 4.5. *The map $\nu_{m'}$ is a well-defined injective group homomorphism, and*

$$\mathrm{w}_{mm'}(\nu_{m'}(x)) = \mathrm{w}_m(x) \prod_{p \in \Delta} (p-1) \tag{4.2}$$

for all $x \in \mathbb{Z}/m\mathbb{Z}$. In particular, if Δ is empty, then $\mathrm{w}_{mm'}(\nu_{m'}(x)) = \mathrm{w}_m(x)$ for all $x \in \mathbb{Z}/m\mathbb{Z}$. If m and m' are relatively prime, then $\mathrm{w}_{mm'}(\nu_{m'}(x)) = \zeta_{m'} \mathrm{w}_m(x)$ for all $x \in \mathbb{Z}/m\mathbb{Z}$.

Proof. The homomorphism claims are exercises. Factor m and m' into primes:

$$m = p_1^{a_1} \cdots p_k^{a_k}, \qquad m' = p_1^{b_1} \cdots p_\ell^{b_\ell},$$

where $\ell \geq k$, $a_i \geq 1$, $b_i \geq 0$ for $1 \leq i \leq k$, and $b_j \geq 1$ for $k+1 \leq j \leq \ell$. The set Δ equals $\{p_{k+1}, \dots, p_\ell\}$, and

$$mm' = p_1^{a_1+b_1} \cdots p_k^{a_k+b_k} p_{k+1}^{b_k} \cdots p_\ell^{b_\ell}.$$

Refer to (4.1). Then $\zeta_m = \prod_{i=1}^{k}(p_i - 1)$, and $\zeta_{mm'} = \prod_{i=1}^{\ell}(p_i - 1)$. The socle of $\mathbb{Z}/m\mathbb{Z}$ is generated by $p_1^{a_1-1} \cdots p_k^{a_k-1}$, while the socle of $\mathbb{Z}/mm'\mathbb{Z}$ is generated by $p_1^{a_1+b_1-1} \cdots p_k^{a_k+b_k-1} p_{k+1}^{b_{k+1}-1} \cdots p_\ell^{b_\ell-1}$. If $x \in \mathrm{soc}(\mathbb{Z}/m\mathbb{Z})$, then $m'x \in \mathrm{soc}(\mathbb{Z}/mm'\mathbb{Z})$. Moreover, if $x = u(m/f)$, then $m'x = u(mm'/f)$. Viewed over $\mathbb{Z}/mm'\mathbb{Z}$, $\bar{\delta}_f$ contains $\{k+1, \dots, \ell\}$. Then the expression for $\mathrm{w}_{mm'}(\nu_{m'}(x))$ in (4.1) is $\prod_{p\in\Delta}(p-1)$ times the expression for $\mathrm{w}_m(x)$.

If $x \notin \mathrm{soc}(\mathbb{Z}/m\mathbb{Z})$, then $m'x \notin \mathrm{soc}(\mathbb{Z}/mm'\mathbb{Z})$. Then $\mathrm{w}_{mm'}(\nu_{m'}(x)) = \zeta_{mm'} = \zeta_m \prod_{p\in\Delta}(p-1) = \mathrm{w}_m(x) \prod_{p\in\Delta}(p-1)$, as desired. □

Suppose a linear code C over $\mathbb{Z}/m\mathbb{Z}$ has a generator matrix G. The next result describes producing a linear code C' over $\mathbb{Z}/mm'\mathbb{Z}$ of the same length and with essentially the same weight enumerator.

Proposition 4.6. *Let C be a linear code over $\mathbb{Z}/m\mathbb{Z}$ with a generator matrix G. Define G' to be the matrix over $\mathbb{Z}/mm'\mathbb{Z}$ obtained by applying $\nu_{m'}$ to every entry of G, and define C' to be the linear code over $\mathbb{Z}/mm'\mathbb{Z}$ generated by G'. Then $|C'| = |C|$, and all weights multiply by $N = \prod_{p\in\Delta}(p-1)$, so that*

$$A_w(C') = \begin{cases} 0, & N \nmid w, \\ A_{w/N}(C), & N \mid w. \end{cases}$$

Proof. Any element of $m\mathbb{Z}/mm'\mathbb{Z}$ annihilates every entry of G'. Thus the coefficients used to form codewords in C' reduce to $(\mathbb{Z}/mm'\mathbb{Z})/(m\mathbb{Z}/mm'\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$. From this, $|C'| = |C|$ follows. The weight formulas follow from (4.2). □

5. Two prime powers

In this section, we examine the cases where $m = p^a q^b$ with primes $p < q$. We will begin with $m = pq$ and build up from there.

Let $m = pq$ for distinct primes $p < q$. Choosing $\zeta = (p-1)(q-1)$, the homogeneous weight w on $\mathbb{Z}/m\mathbb{Z}$ is constant on the four orbits of $\mathcal{U}_m$, as in the table that follows. We also include the sizes of the orbits.

orbit	orb(0)	orb(1)	orb(p)	orb(q)
w	0	$pq-p-q$	$pq-q$	$pq-p$
size	1	$(p-1)(q-1)$	$q-1$	$p-1$

(5.1)

Remark 5.1. One of the first difficulties we face is that, when $p > 2$, we cannot find failures of the MacWilliams identities for the homogeneous weight enumerator by using generator matrices with only one row. Let G be such a generator matrix, size $1 \times n$, with at least one entry being a unit. Let C be the linear code over $\mathbb{Z}/m\mathbb{Z}$ generated by G. Because w is constant on orbits of $\mathcal{U}_m$, we see that

$$\begin{aligned}\mathrm{howe}_C &= 1 + |\mathrm{orb}(1)| t^{\mathrm{w}(G)} + |\mathrm{orb}(p)| t^{\mathrm{w}(pG)} + |\mathrm{orb}(q)| t^{\mathrm{w}(qG)} \\ &= 1 + (p-1)(q-1) t^{\mathrm{w}(G)} + (q-1) t^{\mathrm{w}(pG)} + (p-1) t^{\mathrm{w}(qG)}.\end{aligned}$$

When $p > 2$, the four orbits all have different sizes. The weight enumerator then determines the weights $\mathrm{w}(xG)$ as a function of x. By the extension theorem for w [6, Theorem 2.5], this determines the matrix G, up to monomial equivalence. Equivalent codes have equivalent dual codes with equal homogeneous weight enumerators.

Remark 5.2. When $p = 2$, the orbits orb(1) and orb(p) have the same size. This allows for examples of linear codes, such as those with $m = 6$ in [1, Example 5.2], that have the same homogeneous weight enumerator because they interchange the weights supported on those two orbits. There are similar examples for all $m = 2^a q^b$.

Indeed, let G_{11} and G_{12} be $1 \times q$ matrices over $\mathbb{Z}/2^a q^b\mathbb{Z}$, where G_{11} has all q of its entries equal to q^{b-1}, and G_{12} has $q-2$ entries equal to q^{b-1} and the remaining 2 entries equal to q^b. If C_{11} and C_{12} are the linear codes generated by G_{11} and G_{12} over $\mathbb{Z}/2^a q^b\mathbb{Z}$, then $\mathrm{howe}_{C_{11}} = \mathrm{howe}_{C_{12}} = 1 + (q-1)t^{q(q-2)} + 2q(2^{a-1}-1)t^{q(q-1)} + (q-1)t^{q^2} + t^{2q(q-1)}$. However, $\mathrm{howe}_{C_{11}^\perp} \neq \mathrm{howe}_{C_{12}^\perp}$. The following table shows the smallest weights at which the numbers of dual codewords differ.

Code	q	b	A_{q-1}	A_q
$C_{11}^\perp$	3	1	6	
$C_{12}^\perp$			4	
$C_{11}^\perp$	3	≥ 2	$3^b + 3$	
$C_{12}^\perp$			$7 \cdot 3^{b-1} + 3$	
$C_{11}^\perp$	≥ 5	1	0	0
$C_{12}^\perp$			0	$2q-2$
$C_{11}^\perp$	≥ 5	≥ 2	$q^b - q^2$	
$C_{12}^\perp$			$3q^b - 2q^{b-1} - q^2$	

These examples are not needed for our main result, so we omit the details.

For later use, we make explicit a ring isomorphism $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/pq\mathbb{Z}$, for primes $p < q$. (This is a simple version of the Chinese remainder theorem.)

Let $p < q$ be two primes. Because they are relatively prime, there exist integers s, t so that $sp + tq = 1$. Set $E_1 = tq$ and $E_2 = sp$. Define two maps

$$\begin{aligned} \phi &: \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/pq\mathbb{Z}, \quad \phi(x,y) = xE_1 + yE_2 \bmod pq, \\ \psi &: \mathbb{Z}/pq\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}, \qquad \psi(z) = (z \bmod p, z \bmod q). \end{aligned} \tag{5.2}$$

Proposition 5.3. *Assume $p < q$ are primes. Viewed as elements of $\mathbb{Z}/pq\mathbb{Z}$, E_1 and E_2 are orthogonal idempotents, i.e., $E_1^2 = E_1$, $E_2^2 = E_2$, $E_1E_2 = 0$, and $E_1 + E_2 = 1$. Moreover, $pE_1 = 0$, $qE_2 = 0$, $E_1 \in \operatorname{orb}(q)$, $E_2 \in \operatorname{orb}(p)$, $\mathrm{w}(E_1) = \mathrm{w}(q)$, and $\mathrm{w}(E_2) = \mathrm{w}(p)$. The maps ϕ, ψ of (5.2) are ring isomorphisms, with $\psi = \phi^{-1}$.*

Proof. For example, we verify that $E_1^2 = E_1$, using $sp + tq = 1$:

$$E_1^2 - E_1 = (1 - sp)^2 - (1 - sp) = -sp(1 - sp) = -sptq = 0.$$

The other claims are similar or are routine verifications. □

Continue to assume that $R = \mathbb{Z}/m\mathbb{Z}$, with $m = pq$ for primes $p < q$. The group of units $\mathcal{U}_m$ acts by scalar multiplication on the R-module $M = R \oplus RE_2 = R \oplus pR$. Viewed as column matrices, representatives of the $\mathcal{U}_m$-orbits fall into the four groupings listed below. The groupings are distinguished by the greatest common divisor of the entries of the vectors: 0, 1, E_1, and E_2, respectively.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ E_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2E_2 \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ (q-1)E_2 \end{bmatrix}; \tag{5.3}$$

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 0 \\ E_2 \end{bmatrix}, \begin{bmatrix} E_2 \\ 0 \end{bmatrix}, \begin{bmatrix} E_2 \\ E_2 \end{bmatrix}, \begin{bmatrix} E_2 \\ 2E_2 \end{bmatrix}, \dots, \begin{bmatrix} E_2 \\ (q-1)E_2 \end{bmatrix}. \tag{5.4}$$

The orbit of the zero vector has size 1. Call the next $q+1$ vectors in (5.3) *grouping I*; the orbits generated by each vector in grouping I have size $(p-1)(q-1) = |\mathcal{U}_m|$. The first vector in (5.4), *grouping II*, has an orbit of size $p-1$, while the remaining $q+1$ vectors in (5.4), *grouping III*, each generate orbits of size $q-1$. This is summarized in the following table.

grouping	0	I	II	III
number	1	$q+1$	1	$q+1$
size of orbits	1	$(p-1)(q-1)$	$p-1$	$q-1$

(5.5)

Observe the patterns obtained when grouping I is multiplied by one of the idempotents: by E_1, yielding grouping II; by E_2, yielding grouping III, in order. Note that E_2 annihilates grouping II and E_1 annihilates grouping III.

We will construct linear codes over $\mathbb{Z}/m\mathbb{Z}$ using generator matrices with columns coming from the nonzero vectors in (5.3) and (5.4). A generator matrix G will be specified by a function η, a *multiplicity function*, that records the number of times each nonzero vector in (5.3) and (5.4) appears as a column of G. Multiplying a multiplicity function by a scalar will be called *replication*; replication concatenates copies of G. When η is multiplied by c, all weights are also multiplied by c. If C is the linear code over $\mathbb{Z}/pq\mathbb{Z}$ generated by G, then every codeword of C has the form $[x\ y]G$. Notice that $[0\ E_1]G = 0$, so we may assume $y \in RE_2$. Recall that w is constant on $\mathcal{U}_m$-orbits, so that $\mathrm{w}(u[x\ y]G) = \mathrm{w}([x\ y]G)$ for all $u \in \mathcal{U}_m$. Thus, in order to calculate howe_C, it suffices to know $\mathrm{w}([x\ y]G)$ as $[x\ y]^\top$ varies over the vectors in (5.3) and (5.4). To be specific, let W be the $(2q+3) \times (2q+3)$ matrix whose rows and columns are indexed by the nonzero vectors in (5.3) and (5.4) and whose entry at position (v_i, v_j) is $\mathrm{w}(v_i^\top v_j)$. Similarly, write the multiplicity function η as a $(2q+3) \times 1$ column vector. We then have

$$\mathrm{w}(v_i G) = \sum_j \mathrm{w}(v_i^\top v_j)\eta(v_j), \tag{5.6}$$

which is the v_i-entry of $W\eta$. Except for degenerate cases (when $[x\ y] \mapsto [x\ y]G$ is not injective), the weight enumerator of C is

$$\mathrm{howe}_C = \sum_i |\mathrm{orb}(v_i)| t^{\mathrm{w}(v_i G)}. \tag{5.7}$$

Remark 5.4. The matrix W has appeared in different guises in [2, 5, 9, 13].

Our next objective is to prove that the matrix W is invertible. With that in mind, let P be a permutation matrix of size $(q+1) \times (q+1)$; i.e., P is an invertible integer matrix with entries from $\{0, 1\}$ having exactly one 1 in each row and each column. Then set $P(s,t) = sJ + (t-s)P$, where J is the all-1 matrix; $P(s,t)$ is a $(q+1) \times (q+1)$ matrix with every entry equal to s *except* that each row and column contains exactly one t (in positions given by P). The notation $\mathrm{col}(s)$ (resp. $\mathrm{row}(s)$) means a column vector (resp. row vector) of size $q+1$ with every entry equal to s.

Proposition 5.5. *There exists a permutation matrix P so that the W-matrix for the homogeneous weight with $M = R \oplus RE_2 = R \oplus pR$ has the form*

$$W = \begin{bmatrix} P(\mathrm{w}(1), \mathrm{w}(q)) & \mathrm{col}(\mathrm{w}(q)) & P(\mathrm{w}(p), 0) \\ \mathrm{row}(\mathrm{w}(q)) & \mathrm{w}(q) & \mathrm{row}(0) \\ P(\mathrm{w}(p), 0) & \mathrm{col}(0) & P(\mathrm{w}(p), 0) \end{bmatrix}.$$

The matrix P has a 1 *in positions* $(1,2)$ *and* $(2,1)$. *The matrices P and W are symmetric.*

We emphasize that the same P is used four times in W.

Proof. One must determine the homogeneous weights of the dot products of the nonzero vectors in (5.3) and (5.4). Fix a vector v from grouping I. Its dot product with another vector v' from grouping I will be a unit *with exactly one exception.* For example, if $v = [1, iE_2]^\top$ and $v' = [1, jE_2]^\top$ with $i \neq 0$, their dot product is $1 + ijE_2 = E_1 + (1+ij)E_2$. Because the multiples of E_2 form a copy of the field $\mathbb{Z}/q\mathbb{Z}$, there is exactly one value of j with $1 + ij = 0$, namely $j = -i^{-1}$. In that case, the dot product equals E_1, and $\mathrm{w}(E_1) = \mathrm{w}(q)$. Units have weight $\mathrm{w}(1)$. The vectors $[E_1, E_2]^\top$ and $[1, 0]^\top$ also pair to E_1, handling the remaining cases in grouping I. This means that the permutation matrix P has a 1 in positions $(1,2)$ and $(2,1)$.

The vector $[E_1, 0]^\top$ of grouping II has dot product equal to E_1 when paired with any vector from groupings I or II; it has dot product 0 when paired with any vector from grouping III.

When a fixed vector from grouping III is paired with grouping I or III all but one dot product will be a unit multiple of E_2; the remaining value will be 0. The same argument involving $1+ij=0$ occurs, so that the same permutation matrix is being used four times.

The matrices are symmetric because the ring R is commutative. ☐

Remark 5.6. We reiterate a comment made in the proof above: the permutation matrix P has a 1 in positions $(1,2)$ and $(2,1)$. In particular, this means the rest of the entries in row 2 are 0. Consequently, row 2 of $P(s,t)$ consists of a t in column 1, followed by q entries equal to s.

Proposition 5.7. *The matrix W is invertible, and*

$$W^{-1} = \begin{bmatrix} P(-a,b) & \operatorname{col}(a) & P(a,-b) \\ \operatorname{row}(a) & c & \operatorname{row}(-a) \\ P(a,-b) & \operatorname{col}(-a) & P(d,e) \end{bmatrix},$$

with the same permutation matrix P as in Proposition 5.5*, and with*

$$a = \frac{1}{pq^2}, \quad b = \frac{q-1}{pq^2}, \quad c = \frac{1}{(q-1)pq^2}, \quad d = \frac{1}{(p-1)pq^2}, \quad e = \frac{q-1}{(p-1)pq^2}.$$

Proof. A computation shows that $WW^{-1} = I$. ☐

Define a matrix G_{13} of size $2 \times (2q+3)$ whose columns consist of all the nonzero vectors in (5.3) and (5.4), each appearing once. The multiplicity function η_{13} is the all-1 vector. Let C_{13} be the linear code generated by G_{13} over $R = \mathbb{Z}/pq\mathbb{Z}$.

Proposition 5.8. *The linear code C_{13} is isomorphic to $R \oplus pR$, and $|C_{13}| = pq^2$. The weights of nonzero elements are*

$$\mathrm{w}(vG_{13}) = \begin{cases} 2pq^2 - 2q^2 + pq - 2p, & \gcd(v) = 1, \\ pq^2 + pq - 2p, & \gcd(v) = q, \\ 2pq^2 - 2q^2, & \gcd(v) = p. \end{cases}$$

Proof. Using (5.6) and Proposition 5.5, the weights of nonzero elements are determined by $W\eta_{13}$:

$$\mathrm{w}(xG_{13}) = \begin{cases} q\mathrm{w}(1) + \mathrm{w}(q) + \mathrm{w}(q) + q\mathrm{w}(p), & \gcd(x) = 1, \\ (q+1)\mathrm{w}(q) + \mathrm{w}(q), & \gcd(x) = q, \\ q\mathrm{w}(p) + q\mathrm{w}(p), & \gcd(x) = p. \end{cases}$$

Using (5.1), the weight formulas then simplify to those stated. ☐

Refer to the weights listed in Proposition 5.8 as α, β, γ, respectively:

$$\begin{aligned} \alpha &= 2pq^2 - 2q^2 + pq - 2p, \\ \beta &= pq^2 + pq - 2p, \\ \gamma &= 2pq^2 - 2q^2. \end{aligned} \tag{5.8}$$

Observe that $\beta < \gamma < \alpha$ when p is odd, and $\gamma < \alpha = \beta$ when $p = 2$. Recall that the orbits of vectors have different sizes when p is odd, depending on the gcd of the vector, (5.5).

Corollary 5.9. *The linear code C_{13} has homogeneous weight enumerator*

$$\mathrm{howe}_{C_{13}} = 1 + (p-1)t^{\beta} + (q^2-1)t^{\gamma} + (p-1)(q^2-1)t^{\alpha}$$

when p is odd, and, when $p = 2$, the enumerator is

$$\mathrm{howe}_{C_{13}} = 1 + (q^2-1)t^{\gamma} + q^2 t^{\alpha}.$$

We will use the linear code C_{13} and W^{-1} to define two linear codes with the same homogeneous weight enumerator but whose dual codes have different weight enumerators. The list of weights of $\mathcal{U}$-orbits of codewords is given by $W\eta$, as in (5.6), where η is the multiplicity function of the code. The code C_{13} of Proposition 5.8 has η_{13} equal to the all-1 vector, and the list of weights is the column vector $w = \langle \alpha, \ldots, \alpha; \beta; \gamma, \ldots, \gamma \rangle$, where there are $q + 1$ of both α and γ. Making use of the different sizes of orbits, (5.5), we create a new list of weights:

$$w' = \langle \gamma, \alpha, \ldots, \alpha; \beta; \gamma, \ldots, \gamma, \alpha, \ldots, \alpha \rangle,$$

consisting of one γ followed by q α's in grouping I, then one β in grouping II, and finishing with $q - p + 2$ γ's and $p - 1$ α's in grouping III. Taking the different sizes of orbits into account, w and w' yield the same weight enumerator, as in (5.7).

We will show that there exists a linear code D_{13} that achieves this list w' of weights, at least up to a replication factor. By replicating both C_{13} and D_{13} in the same way (call the results C_{14} and D_{14}), we achieve, by design, $\mathrm{howe}_{D_{14}} = \mathrm{howe}_{C_{14}}$. We will also show that $A_{\mathrm{w}(p)}(D_{14}^{\perp}) \neq A_{\mathrm{w}(p)}(C_{14}^{\perp})$ when $pq \neq 6$ and that $A_2(D_{14}^{\perp}) \neq A_2(C_{14}^{\perp})$ when $pq = 6$.

Remark 5.10. We call attention to one technical detail. In w', the $p - 1$ α's occupy the last positions in grouping III. In the matrix $P(s, t)$ of Proposition 5.5, the second row has a t in the first column. This means that the last columns of grouping III in that same row all have entries equal to s. See Remark 5.6.

Lemma 5.11. *The entries of $W^{-1}w'$ are positive rational numbers.*

Proof. The form of the entries depends on the location of the exceptional entries of $P(s, t)$: Is the initial γ in w' multiplied by s or by t? Is there a t in the last $p - 1$ columns or not? The answers to these questions lead to four possibilities of the form of the entries in grouping I and, similarly, four possibilities in grouping III. However, Remark 5.10 says that one of the four possibilities is prohibited (marked with $*$). The formats, together with the number d of times they appear, are in Table 5.1, with the entry for grouping II marked with †. Because $2 \leq p < q$, all of these expressions are positive rational numbers, with the exception of the very first—which is prohibited from occurring. ☐

TABLE 5.1. Formats of entries in $W^{-1}w'$

$$\begin{aligned}
(*)\quad & b\gamma - qa\alpha + a\beta + (q-p+2)a\gamma + (p-2)a\alpha - b\alpha \\
&\qquad = (pq - q^2 - 2p + 4q)/q^2, \quad d = 0, \\
& b\gamma - qa\alpha + a\beta + (q-p+1)a\gamma - b\gamma + (p-1)a\alpha \\
&\qquad = (pq - 2p + 2q)/q^2, \quad d = 1, \\
-a\gamma + b\alpha - (q-1)a\alpha + {}& a\beta + (q-p+2)a\gamma + (p-2)a\alpha - b\alpha \\
&\qquad = (pq - 2p + 2q)/q^2, \quad d = p-1, \\
-a\gamma + b\alpha - (q-1)a\alpha + {}& a\beta + (q-p+1)a\gamma - b\gamma + (p-1)a\alpha \\
&\qquad = (pq + q^2 - 2p)/q^2, \quad d = q-p+1;
\end{aligned}$$

$$\begin{aligned}
(\dagger)\quad & a\gamma + qa\alpha + c\beta - (q-p+2)a\gamma - (p-1)a\alpha \\
&\qquad = (q^2 - pq + 2p)/q^2, \quad d = 1;
\end{aligned}$$

$$\begin{aligned}
(*)\quad & -b\gamma + qa\alpha - a\beta + (q-p+2)d\gamma + (p-2)d\alpha - e\alpha \\
&\qquad = (2pq - 2p - 3q + 4)/(p-1)q, \quad d = 0, \\
& -b\gamma + qa\alpha - a\beta + (q-p+1)d\gamma - e\gamma + (p-1)d\alpha \\
&\qquad = 2(q-1)/q, \quad d = 1, \\
a\gamma - b\alpha + (q-1)a\alpha - {}& a\beta + (q-p+2)d\gamma + (p-2)d\alpha - e\alpha \\
&\qquad = (pq - 2q + 2)/(p-1)q, \quad d = p-1, \\
a\gamma - b\alpha + (q-1)a\alpha - {}& a\beta + (q-p+1)d\gamma - e\gamma + (p-1)d\alpha \\
&\qquad = 1, \quad d = q-p+1.
\end{aligned}$$

We now define a replicated version C_{14} of the linear code C_{13} of Proposition 5.8, as well as a linear code D_{14}. Note that the least common multiple of the denominators appearing in Table 5.1 is $(p-1)q^2$.

Definition 5.12. Define the multiplicity function $\eta_{C_{14}}$ of C_{14} to be the vector of length $2q+3$ all of whose entries equal $(p-1)q^2$; C_{14} is a replicated version of C_{13}. Define the multiplicity function $\eta_{D_{14}}$ of D_{14} to be the vector of length $2q+3$ equal to $(p-1)q^2W^{-1}w'$; by Lemma 5.11, all the entries of $\eta_{D_{14}}$ are positive integers.

Remark 5.13. We record for later use the number of columns in generator matrices for C_{14} and D_{14} from the three groupings in (5.3) and (5.4). These counts use the formulas and multiplicities d found in Table 5.1, together with the replication factor $(p-1)q^2$.

Grouping	C_{14}	D_{14}
I	$(q+1)(p-1)q^2$	$(p-1)(q^3+q^2+pq-2p)$
II	$(p-1)q^2$	$(p-1)(q^2-pq+2p)$
III	$(q+1)(p-1)q^2$	$(q+1)(p-1)q^2$

(5.9)

Note that the counts for grouping III are equal and that the totals over all three groupings are also equal (so the codes have the same length).

Before we analyze dual codewords of small weight, we isolate a situation that will occur often.

Lemma 5.14. *Suppose C is a linear code of length n over $\mathbb{Z}/m\mathbb{Z}$ with generator matrix G. Fix a $\mathcal{U}_m$-orbit $\operatorname{orb}(x)$, $x \in \mathbb{Z}/m\mathbb{Z}$. If scalar multiplication by x annihilates $N(x)$ columns of G, then singletons with nonzero entry in $\operatorname{orb}(x)$ contribute $|\operatorname{orb}(x)| \cdot N(x)$ to $A_{\mathrm{w}(x)}(C^\perp)$.*

Proof. Consider a singleton v with nonzero entry ux, for some $u \in \mathcal{U}_m$. Then $\mathrm{w}(v) = \mathrm{w}(x)$, and $v \in C^\perp$ if and only if the position of the nonzero entry of v matches the position of a column of G annihilated by x. □

Recall that α, β, γ are defined in (5.8).

Theorem 5.15. *For $m = pq$, $p < q$ prime, the linear codes C_{14}, D_{14} over $\mathbb{Z}/m\mathbb{Z}$ from Definition 5.12 have the same homogeneous weight enumerator and their dual codes have different homogeneous weight enumerators. Specifically, the common weight enumerator for C_{14} and D_{14} is*

$$1 + (p-1)t^{(p-1)q^2\beta} + (q^2-1)t^{(p-1)q^2\gamma} + (p-1)(q^2-1)t^{(p-1)q^2\alpha}$$

when p is odd, and is

$$1 + (q^2-1)t^{(p-1)q^2\gamma} + q^2 t^{(p-1)q^2\alpha}$$

when $p = 2$. Moreover, $A_{\mathrm{w}(p)}(C_{14}^\perp) \neq A_{\mathrm{w}(p)}(D_{14}^\perp)$ for $m \neq 6$, and $A_2(C_{14}^\perp) \neq A_2(D_{14}^\perp)$ when $m = 6$.

Proof. By construction and the choice of w', C_{14} and D_{14} have the same homogeneous weight enumerator. The format follows from Corollary 5.9, with all the weights multiplied by the replication factor $(p-1)q^2$.

Note from (5.1) that $\mathrm{w}(1) < \mathrm{w}(p) < \mathrm{w}(q)$. Except for $m = 6$, i.e., $p = 2$, $q = 3$, we also have $\mathrm{w}(p) < 2\mathrm{w}(1)$; this implies that a vector v with $\mathrm{w}(v) = \mathrm{w}(p)$ must be a singleton whose nonzero entry is a unit multiple of p. (When $m = 6$, $2\mathrm{w}(1) = 2 < 3 = \mathrm{w}(2)$: another 'curse of small values.') See Remark 4.4.

The column in grouping II is annihilated by p and its unit multiples. Thus

$$A_{\mathrm{w}(p)}(C_{14}^{\perp}) = (q-1)(p-1)q^2, \tag{5.10}$$

$$A_{\mathrm{w}(p)}(D_{14}^{\perp}) = (q-1)(p-1)(q^2 - pq + 2p), \tag{5.11}$$

using the (†) expression in Table 5.1 and (5.9). Because $2 \leq p < q$, $A_{\mathrm{w}(p)}(C_{14}^{\perp}) \neq A_{\mathrm{w}(p)}(D_{14}^{\perp})$ for $m \neq 6$.

When $m = 6$, choose $\zeta = 2$, so that w has the following values:

x	0	± 1	± 2	3
$\mathrm{w}(x)$	0	1	3	4

There are no dual codewords of weight 1 because there are no zero-columns in the generator matrices. The only vectors having weight 2 are doubletons with nonzero entries equaling ± 1. The columns in (5.3) and (5.4) represent different $\mathcal{U}_m$-orbits, so the only way for columns to be ± 1 multiples of each other is for the columns to be in the same orbit. Moreover, only the column in grouping II has order two. For columns from grouping II, all four signs in $(\pm 1, \pm 1)$ are possible; for all other columns, only two choices of signs, $\pm(1, -1)$, are possible. Thus, a linear code C with multiplicity function η has

$$A_2(C^{\perp}) = 4\binom{\eta(\mathrm{II})}{2} + 2\sum_{v \in \mathrm{I} \cup \mathrm{III}} \binom{\eta(v)}{2}.$$

For our particular codes, a calculation gives $A_2(C_{14}^{\perp}) = 720$ and $A_2(D_{14}^{\perp}) = 722$. □

We now generalize Theorem 5.15 to the case where $m = p^a q^b$ for primes $p < q$, $a, b \geq 1$. As in (4.1) and (5.1), we choose $\zeta = (p-1)(q-1)$, so that the homogeneous weight on $\mathbb{Z}/p^a q^b \mathbb{Z}$ has the following values.

orbit	w
$\mathrm{orb}(0)$	0
$\mathrm{orb}(p^{a-1}q^{b-1})$	$pq - p - q$
$\mathrm{orb}(p^a q^{b-1})$	$pq - q$
$\mathrm{orb}(p^{a-1}q^b)$	$pq - p$
others	$pq - p - q + 1$

The ‘others’ subset is empty when $a = b = 1$, by Remark 4.2.

We now define two linear codes C_{15}, D_{15} over $\mathbb{Z}/p^a q^b \mathbb{Z}$. The generator matrices of C_{15}, D_{15} are obtained from the generator matrices of C_{14}, D_{14} (whose columns have the form given in (5.3) and (5.4)) by applying Proposition 4.6 with $m' = p^{a-1}q^{b-1}$. Recall that α, β, γ are defined in (5.8).

Theorem 5.16. *The linear codes C_{15}, D_{15} over $\mathbb{Z}/m\mathbb{Z}$, $m = p^a q^b$, with primes $p < q$, satisfy $\mathrm{howe}_{C_{15}} = \mathrm{howe}_{D_{15}}$ and $\mathrm{howe}_{C_{15}^{\perp}} \neq \mathrm{howe}_{D_{15}^{\perp}}$. Specifically, the common weight enumerator for C_{15} and D_{15} is*

$$1 + (p-1)t^{(p-1)q^2\beta} + (q^2-1)t^{(p-1)q^2\gamma} + (p-1)(q^2-1)t^{(p-1)q^2\alpha}$$

when p is odd, and is

$$1 + (q^2 - 1)t^{(p-1)q^2\gamma} + q^2 t^{(p-1)q^2\alpha}$$

when $p = 2$.

The following table shows the smallest weights at which the numbers of dual codewords differ; T_1, T_2, T_3 are non-zero constants depending on m that are explained in the proof.

Code	*pq*	*a*	*b*	A_{pq-p-q}	$A_{pq-p-q+1}$	A_{pq-q}
$C_{15}^{\perp}$	6	1	1	0	720	
$D_{15}^{\perp}$				0	722	
$C_{15}^{\perp}$	$\neq 6$	1	1	0	0	(5.10)
$D_{15}^{\perp}$				0	0	(5.11)
$C_{15}^{\perp}$	*all*	≥ 2	1	$(p-1)\cdot(5.10)$		
$D_{15}^{\perp}$				$(p-1)\cdot(5.11)$		
$C_{15}^{\perp}$	*all*	≥ 1	≥ 2	T_1	$T_2 + T_3 \cdot (5.10)$	
$D_{15}^{\perp}$				T_1	$T_2 + T_3 \cdot (5.11)$	

Proof. By Lemma 4.5 and Proposition 4.6, the homogeneous weight enumerators of C_{15}, D_{15} are exactly the same as those of C_{14}, D_{14}, hence equal.

Now examine the dual codes. As in the proof of Theorem 5.15, we need to distinguish the case where $pq = 6$, i.e., $p = 2$, $q = 3$, from the cases where $pq \neq 6$. We begin with $pq \neq 6$.

When $pq \neq 6$, the following inequalities hold:

$$pq - p - q < pq - p - q + 1 < pq - q < 2(pq - p - q).$$

Thus, in order for a vector to have weight $pq - p - q$, $pq - p - q + 1$, or $pq - q$, the vector must be a singleton whose nonzero entry is, respectively, a unit multiple of $p^{a-1}q^{b-1}$, an element not in the socle of $\mathbb{Z}/m\mathbb{Z}$, or a unit multiple of $p^a q^{b-1}$. In addition, in order for a singleton to be in the dual code, its nonzero entry must annihilate the corresponding column of the generator matrix. Any such column, of course, is of the form $\nu_{m'}$ applied to a column from (5.3) and (5.4). We will call these new columns 'm'-scaled columns.'

When $a = b = 1$, we are in the situation of Theorem 5.15, so the counts for $A_{pq-q} = A_{\mathrm{w}(p)}$ match those in (5.10) and (5.11).

When $a \geq 2$, $b = 1$, the orbit of $p^{a-1}q^{b-1} = p^{a-1}$ annihilates m'-scaled columns from grouping II only. The size of the orbit is $(p-1)(q-1)$, so the counts for A_{pq-p-q} are $p - 1$ times those in (5.10) and (5.11).

When $a = 1$, $b \geq 2$, the orbit of $p^{a-1}q^{b-1} = q^{b-1}$ annihilates m'-scaled columns from grouping III only. Since C_{15} and D_{15} have the same number of columns of grouping III (5.9), the contribution to A_{pq-p-q} will be the same (call it T_1). Of the elements not in the socle of $\mathbb{Z}/m\mathbb{Z}$, the units do not annihilate any columns, and the unit multiples of p annihilate m'-scaled columns of grouping II only (contributing a term of the form $T_3\eta(\text{II})$ to $A_{pq-p-q+1}$ of the dual codes). The unit multiples of $q, q^2, \ldots, q^{b-2}$ annihilate m'-scaled columns of grouping III only, and the remaining elements not in the socle of $\mathbb{Z}/m\mathbb{Z}$ annihilate m'-scaled columns of all groupings. These last two contributions are the same for both codes (call it T_2), because C_{15} and D_{15} have the same number of columns of grouping III and the same total number of columns (5.9).

When $a, b \geq 2$, $p^{a-1}q^{b-1}$ annihilates every m'-scaled column in (5.3) and (5.4). Since C_{15} and D_{15} have the same total number of columns (5.9), the contribution to A_{pq-p-q} will be the same (again, call it T_1). Elements of $\mathbb{Z}/m\mathbb{Z}$ that are not in the socle are in the orbits of $p^i q^j$ with $i < a-1$ or $j < b-1$. Those in the orbits of $p, p^2, \ldots, p^a$ (i.e., $i \geq 1$ and $j = 0$) annihilate m'-scaled columns in grouping II only. Setting $T_3 = (\sum_{i=1}^{a} |\text{orb}(p^i)|)/(q-1)$, these elements contribute T_3 times (5.10) and (5.11), resp., to $A_{pq-p-q+1}$. The orbits of $q, q^2, \ldots, q^b$ (i.e., $i = 0$ and $j \geq 1$) annihilate m'-scaled columns from grouping III only, and the remaining orbits not in the socle annihilate all m'-scaled columns. From (5.9), C_{15} and D_{15} have the same number of columns from grouping III as well as the same total number of columns; these contributions to $A_{pq-p-q+1}$ are the same, say T_2.

We now shift to $pq = 6$, i.e., $p = 2$, $q = 3$. When $a = b = 1$, i.e., $m = 6$, Theorem 5.15 applies, and $A_2(C_{15}^{\perp}) = 720$ while $A_2(D_{15}^{\perp}) = 722$. When $a \geq 2$ and $b = 1$, then the unit multiples of $2^{a-1} = 2^{a-1}3^{b-1}$ annihilate the m'-scaled column from grouping II. The counts for $A_1 = A_{pq-p-q}$ are thus those from (5.10) and (5.11): $A_1(C_{15}^{\perp}) = 18$ and $A_1(D_{15}^{\perp}) = 14$.

When $b \geq 2$, unit multiples of $2^{a-1}3^{b-1}$ annihilate m'-scaled columns from grouping III only (when $a = 1$) or from all groupings (when $a \geq 2$). In either case, since C_{15} and D_{15} have the same number of such columns (5.9), the contribution (called T_1) to A_1 of the dual codes will be the same.

Dual codewords of weight 2 come in two forms: singletons with nonzero entry not in the socle of $\mathbb{Z}/m\mathbb{Z}$ that annihilates an m'-scaled column; or doubletons with nonzero entries of the form $\pm 2^{a-1}3^{b-1}, \pm 2^{a-1}3^{b-1}$ so that the corresponding linear combination of columns vanishes. The counts for single entries not in the socle are as above: the contribution is of the form $T_2 + T_3\eta(\text{II})$. For the other situation, first note that scalar multiplying the m'-scaled columns of (5.3) and (5.4) by $2^{a-1}3^{b-1}$ always annihilates grouping III. Groupings I and II are sent to $[3^b, 0]^{\top}$ when $a = 1$, and they are annihilated when $a \geq 2$. Because 3^b has order 2 when $a = 1$, all signs are possible. Since the sum of the number of columns in groupings I and II is the same for C_{15} and D_{15} (5.9), the contributions are the same to $A_2 = A_{pq-p-q+1}$ of the dual codes and can be incorporated into T_2. □

6. Main theorem for homogeneous weight enumerators

In this section we discuss the general case where $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. In outline: we carefully pick two of the primes, say $p = p_i$ and $q = p_j$, and use Theorem 5.15 to produce two linear codes C', D' over $\mathbb{Z}/p_i p_j\mathbb{Z}$ that have the same homogeneous weight enumerator. The generator matrices of C' and D' use columns from (5.3) and (5.4) with $p = p_i$ and $q = p_j$. We then use Proposition 4.6 with $m' = m/(p_i p_j)$

to produce codes C, D over $\mathbb{Z}/m\mathbb{Z}$ with the same homogeneous weight enumerator. The generator matrices of C and D use m'-scaled columns. Finally, we analyze dual codewords of small weight to show that the dual codes of C and D have different weight enumerators.

Lemma 6.1. *Let $x \in \mathbb{Z}/m\mathbb{Z}$. Then x annihilates m'-scaled columns in groupings I, II, and III according to the chart below.*

case	$x \in (p_i)$*?*	$x \in (p_j)$*?*	x *annihilates*
1	*no*	*no*	*none*
2	*no*	*yes*	*III only*
3	*yes*	*yes*	*all*
4	*yes*	*no*	*II only*

Proof. In (5.3) and (5.4), with $p = p_i$ and $q = p_j$, we have that E_1 is a unit multiple of p_j and E_2 is a unit multiple of p_i. Now apply $\nu_{m'}$. The gcd of grouping I m'-scaled columns is $m' = m/(p_i p_j)$; these columns are annihilated by $(p_i p_j)$. The gcd of grouping II m'-scaled columns is $m/(p_i)$; these columns are annihilated by (p_i). The gcd of grouping III m'-scaled columns is $m/(p_j)$; these columns are annihilated by (p_j). □

Theorem 6.2. *Let m be a positive integer greater than* 5 *that is not prime. Then there exist linear codes C and D over $\mathbb{Z}/m\mathbb{Z}$ such that* $\mathrm{howe}_C = \mathrm{howe}_D$ *but* $\mathrm{howe}_{C^\perp} \neq \mathrm{howe}_{D^\perp}$.

Proof. Let $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of m, with primes $p_1 < p_2 < \cdots < p_k$ and exponents $a_i \geq 1$ for $i = 1, 2, \ldots, k$. If $k = 1$, i.e., m is a prime power, then the result follows from Theorem 3.2. If $k = 2$, the result follows from Theorem 5.16. It remains to consider $k \geq 3$, and we treat two cases.

1. Assume $a_j = 1$ for some $j \in \{2, 3, \ldots, k\}$. Set $p = p_1$ and $q = p_j$, where $a_j = 1$. Apply Theorem 5.15 to produce linear codes C' and D' over $\mathbb{Z}/p_1 p_j\mathbb{Z}$, using (5.3) and (5.4), whose homogeneous weight enumerators are equal but whose dual weight enumerators are different. Apply Proposition 4.6 with $m' = m/(p_1 p_j)$ to obtain linear codes C and D over $\mathbb{Z}/m\mathbb{Z}$ generated by the m'-scaled generator matrices of C' and D'. Also by Proposition 4.6, we have $\mathrm{howe}_C = \mathrm{howe}_D$.

Let ℓ be an element of $\{1, 2, \ldots, k\} - \{1, j\}$; such an ℓ exists because $k \geq 3$. Set $x = m/(p_j p_\ell)$. Write $\xi = \mathrm{w}(x)$; Proposition 4.3 implies that $\xi < \zeta$. By Remark 4.4, any vector v with $\mathrm{w}(v) = \xi$ must be a singleton. Note that $x \in (p_1)$ and $x \notin (p_j)$; the latter statement uses the hypothesis that $a_j = 1$. By Lemma 6.1, x annihilates the m'-scaled columns in grouping II only. Any singleton using x or a unit multiple of x in a position with a grouping II column will contribute to $A_\xi(C^\perp)$ and to $A_\xi(D^\perp)$. The numbers of such contributions are proportional to $\eta(\mathrm{II})$ of the two codes, which are different, (5.9).

Let y be any other element of $\mathbb{Z}/m\mathbb{Z}$ with $\mathrm{w}(y) = \xi$. By Lemma 6.1, singleton vectors using y contribute equally to $A_\xi(C^\perp)$ and $A_\xi(D^\perp)$ (in cases 1, 2, and 3 of the lemma) or contribute differently (in case 4 of the lemma) in a manner proportional to $\eta(\mathrm{II})$. Having considered all the singletons of weight ξ, we conclude that $A_\xi(C^\perp) \neq A_\xi(D^\perp)$.

2. *Assume $a_j > 1$ for all $j \in \{2, \dots, k\}$.* Set $p = p_2$ and $q = p_3$, so that $a_2 > 1$ and $a_3 > 1$. Apply Theorem 5.15 to produce linear codes C' and D' over $\mathbb{Z}/p_2p_3\mathbb{Z}$, using (5.3) and (5.4), whose homogeneous weight enumerators are equal but whose dual weight enumerators are different. Apply Proposition 4.6 with $m' = m/(p_2p_3)$ to obtain linear codes C and D over $\mathbb{Z}/m\mathbb{Z}$ generated by the m'-scaled generator matrices of C' and D'. Also by Proposition 4.6, we have $\mathrm{howe}_C = \mathrm{howe}_D$.

Consider $x = p_2$. Note that x is not in $\mathrm{soc}(\mathbb{Z}/m\mathbb{Z})$ because x lacks (at least) a factor of p_3 (as $a_3 > 1$). Then $\mathrm{w}(x) = \zeta$. Any other y that is not in $\mathrm{soc}(\mathbb{Z}/m\mathbb{Z})$ also has $\mathrm{w}(y) = \zeta$. Clearly, $x \in (p_2)$ and $x \notin (p_3)$. By Lemma 6.1, x annihilates m'-scaled columns in grouping II only. Then x contributes differently to $A_\zeta(C^\perp)$ and $A_\zeta(D^\perp)$ in a manner proportional to $\eta(\mathrm{II})$. Other y not in $\mathrm{soc}(\mathbb{Z}/m\mathbb{Z})$ contribute equally to $A_\zeta(C^\perp)$ and $A_\zeta(D^\perp)$ or differently in a manner proportional to $\eta(\mathrm{II})$, according to the cases in Lemma 6.1.

If $6 \nmid m$, any vector v with $\mathrm{w}(v) = \zeta$ must be a singleton, Remark 4.4, so there are no further contributions with weight ζ. If $6 \mid m$, then $p_1 = 2$ and $p_2 = 3$. By Proposition 4.3, unit multiples of $x_0 = m/(p_1p_2) = m/6$ satisfy $2\mathrm{w}(x_0) = \zeta$; thus, doubleton vectors v with $\mathrm{w}(v) = \zeta$ exist. However, $x_0 \in (p_2)$ (as $a_2 > 1$) and $x_0 \in (p_3)$ (as $a_3 \geq 1$); by Lemma 6.1, x_0 annihilates m'-scaled columns of all groupings. Since C and D have the same total number of columns, doubleton vectors v with $\mathrm{w}(v) = \zeta$ make the same contributions to $A_\zeta(C^\perp)$ and $A_\zeta(D^\perp)$.

In total, contributions to $A_\zeta(C^\perp)$ and $A_\zeta(D^\perp)$ have the form $T_1 + T_2\eta(\mathrm{II})$. We conclude that $A_\zeta(C^\perp) \neq A_\zeta(D^\perp)$. □

Example 6.3. Let $m = 504 = 2^3 \cdot 3^2 \cdot 7$. Set $p = 2$ and $q = 7$. Then $E_1 = 7$ and $E_2 = 8$. The generator matrix for the code C'_{16} over $\mathbb{Z}/14\mathbb{Z}$ is

$$G'_{16} = \left[\begin{array}{cccccccc|c|cccccccc} 7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 8 & 0 & 8 & 2 & 10 & 4 & 12 & 6 & 0 & 8 & 0 & 8 & 2 & 10 & 4 & 12 & 6 \end{array}\right].$$

Then $\mathrm{howe}_{C'_{16}} = 1 + 48t^{98} + 49t^{108}$, consistent with Corollary 5.9. Obtain the multiplicity function of the second code D'_{16} via $W^{-1}w'$, as in Definition 5.12. This multiplicity function has denominators with 49 as least common multiple.

Obtain replicated versions of these codes, clearing denominators by multiplying by 49. The multiplicity function of C'_{17} consists of 17 entries of 49, while the multiplicity function of D'_{17} is

$$\left\langle \begin{array}{cccccccc|c|cccccccc} 59 & 24 & 24 & 59 & 59 & 59 & 59 & 59 & 39 & 49 & 84 & 14 & 49 & 49 & 49 & 49 & 49 \end{array} \right\rangle.$$

Then $\mathrm{howe}_{C'_{17}} = \mathrm{howe}_{D'_{17}} = 1 + 48t^{4802} + 49t^{5292}$.

To get linear codes over $\mathbb{Z}/504\mathbb{Z}$, first note that $504 = 14 \cdot 36$. Apply ν_{36} to each entry of G'_{16}, and use the same multiplicity functions as above. Then $\mathrm{howe}_{C_{17}} = \mathrm{howe}_{D_{17}} = 1 + 48t^{9604} + 49t^{10584}$.

With $\zeta = 12$, the values of w on $\mathbb{Z}/504\mathbb{Z}$, by orbits, are

x	0	252	168	84	72	36	24	12	non-socle
$\lvert\mathrm{orb}(x)\rvert$	1	1	2	2	6	6	12	12	462
$\mathrm{w}(x)$	0	24	18	6	14	10	11	13	12

Set $x = 36 = 504/14$. Then $\xi = \mathrm{w}(x) = 10 < \zeta = 12$. The only elements $r \in \mathbb{Z}/504\mathbb{Z}$ with $\mathrm{w}(r) = 10$ are the six unit multiples of x. Those elements annihilate grouping II columns, but no other groupings. Thus $A_{10}(C_{17}^{\perp}) = 6 \cdot 49 = 294$, while $A_{10}(D_{17}^{\perp}) = 6 \cdot 39 = 234$. Similarly, using $y = 24 = 504/21$, one sees that $A_{11}(C_{17}^{\perp}) = 12 \cdot 49 = 588$, while $A_{11}(D_{17}^{\perp}) = 12 \cdot 39 = 468$.

Acknowledgment

This paper is dedicated to the memory of Tadashi Nagano, one of my most influential teachers as an undergraduate. He later encouraged me to study the subgroups of Spin(n), which eventually led me to study algebraic coding theory.

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Received: July 28, 2021
Accepted: November 17, 2021