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EXTINCTION TIME OF AN EPIDEMIC WITH INFECTION-AGE-DEPENDENT INFECTIVITY

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Abstract. This paper studies the distribution function of the time of extinction of a subcritical epidemic, when a large enough proportion of the population has been immunized and/or the infectivity of the infectious individuals has been reduced, so that the effective reproduction number is less than one. We do that for a SIR/SEIR model, where infectious individuals have an infection-age-dependent infectivity, as in the model introduced in Kermack and McKendrick's seminal 1927 paper. Our main conclusion is that simplifying the model as an ODE SIR model, as it is largely done in the epidemics literature, introduces a bias toward shorter extinction time.

1. Introduction

Consider an epidemic that is declining: the number M of infected individuals is moderate and decreasing, while the total population size N is much larger. In such a phase, the approximation by the deterministic model is no longer valid. Rather, as the initial phase of an epidemic, the final phase can be well approximated by a branching process, in this case a subcritical branching process. The extinction time is thus random. It is of interest to have some information on the distribution function of this extinction time. Indeed, if the subcriticality is due in part to some rules imposed to the population, like mask wearing in public transport, classrooms, workplace, theaters, etc., it is important to evaluate how long such rules must be maintained.

Our epidemic model is a SIR/SEIR model, i.e., we assume that after having been infected and having recovered, an individual remains immune to the disease forever. This is not quite realistic. However, if the duration of the studied period is not too long, then the number of individuals who lose their immunity during

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that period can be neglected. On the other hand, the stochastic SIR/SEIR model upon which we base our analysis is non-Markov. Following the ideas of Kermack–McKendrick [11] and Forien, Pang, Pardoux [6], we consider a model where the infectivity of each infectious individual is infection-age-dependent (and random, the realizations corresponding to various individuals being i.i.d., that is, independent and identically distributed). We characterize the distribution function of the extinction time of the approximating non-Markov branching process with a single ancestor as the unique solution of a Volterra-type integral equation, for which we give a converging numerical approximation. The derivation of the equation is based upon a methodology introduced by Crump and Mode [4]. From this result, we deduce in Theorem 3.4 a formula for the time we have to wait after t_0 for the epidemic to go extinct if at time t_0 we have M infected individuals in a population of size N, with $M \ll N$.

With the help of a numerical scheme, we compute an approximation of the distribution function of the time of extinction, and compare the result with the distribution function of the extinction time of a Markov branching process which approximates the classical Markov SIR model (whose law of large numbers limit is the most standard SIR ODE model), which is known explicitly. This comparison is done between two models which have both the same effective reproduction number R_{eff} (the mean number of "descendants" one infectious individual has at this stage of the epidemic), and the same rate ρ of continuous-time exponential decrease. Our conclusion is that the usual ODE SIR model leads to an underestimation of the extinction time.

Our work was inspired by the recent work of Griette et al. [8], where the authors neglect the new infections during the final phase. Note that this approximation is justified by the data, in the case of the end of the COVID epidemic in Wuhan in 2020. Our work does not make such a simplifying assumption, and allows a very general law for the varying infectivity, and a completely arbitrary law for the duration of the infectious period.

The paper is organized as follows. We present our varying infectivity SIR model in Section 2, together with its branching process approximation, and we give a justification of this approximation. In Section 3, we study the distribution function of the extinction time of the branching process. In Section 4, we present several examples of SIR/SEIR models, including the classical ODE SIR model, ODE SEIR model, and we specify the type of varying infectivity which we have in mind. In Section 5, we compare the time of extinction of the branching approximations to our varying infectivity model, and to the ODE SIR model. In Section 6, we discuss the results obtained in that comparison. Finally in Appendix A, we establish the convergence of a numerical approximation scheme of the equation established in Section 3.

Notation. We shall use the following notations. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = [0, \infty)$. For $x \in \mathbb{R}_+$, $[x]$ denotes the integer part of x and $\lceil x \rceil$ (resp., $\lfloor x \rfloor$) denotes the ceiling function (resp., the floor function). For $x \in \mathbb{R}$, x^+ (resp., x^-) denotes the positive (resp., negative) part of x. For

$(a,b) \in \mathbb{R}^2$, $a < b$, $\mathcal{U}([a,b])$ denotes the uniform distribution on $[a,b]$. $D([0,\infty))$ denotes the space of functions from $[0,\infty)$ into $\mathbb{R}$ that are right-continuous and have left limits at any $t > 0$. We shall always equip the space $D([0,\infty))$ with the Skorohod topology, for the definition of which we refer the reader to Billingsley [2].

2. The SIR model with varying infectivity

2.1. **The epidemic model.** Let $\{\lambda_j(t) : t \geq 0\}$, $j \in \mathbb{Z}\backslash\{0\}$, be a collection of mutually independent non-negative functions, which are such that the $\{\lambda_j\}_{j\geq 1}$ are identically distributed, as well as the $\{\lambda_j\}_{j\leq -1}$. We assume that these functions belong a.s. to $D([0,\infty))$. We consider a SIR model which is such that the j-th initially infected individual has infectivity $\lambda_{-j}(t)$ at time t, while the j-th individual infected after time 0 has at time t infectivity $\lambda_j(t-\tau_j)$ if $0 < \tau_1 < \cdots < \tau_\ell < \cdots$ denote the successive times of infection after time 0 in the population. The quantity $t - \tau_j$ is the age of infection of an individual j at time t. Note that we assume that λ_j vanishes on $\mathbb{R}_-$. The newly infected individual is chosen uniformly at random in the population, and if that individual is susceptible, then he/she jumps from the S to the I compartment at its time of infection while nothing happens if the individual is not susceptible. Examples of function $\lambda_j(t)$ will be given below. That function can be first 0 during the exposed period, then the individual becomes infectious, and at age of infection $\eta_j = \sup\{t : \lambda_j(t) > 0\}$, the individual recovers (i.e., jumps into the R compartment) and is immune for ever. Clearly an important quantity is the total force of infection in the population at time t: $\mathfrak{F}^N(t)$, which is the sum of all the infectivities of the infected individuals at that time. Here N is the total number of individuals in the population. The sum of the numbers of individuals in the three compartments is constant in time: $S^N(t) + I^N(t) + R^N(t) = N$ for all $t \geq 0$. For $X = S, \mathfrak{F}, I$ or R, we define the renormalized quantity $\bar{X}^N(t) = X^N(t)/N$. The main result of [6] is that as $N \to \infty$, $(\bar{S}^N(t), \bar{\mathfrak{F}}^N(t), \bar{I}^N(t), \bar{R}^N(t)) \to (\bar{S}(t), \bar{\mathfrak{F}}(t), \bar{I}(t), \bar{R}(t))$ in probability locally uniformly in t, where the limit is the unique solution of the following system of integral equations, which already appears in the seminal paper of Kermack and McKendrick [11]:

$$\begin{cases} \bar{S}(t) = \bar{S}(0) - \displaystyle\int_0^t \bar{S}(s)\bar{\mathfrak{F}}(s)\,ds, \\ \bar{\mathfrak{F}}(t) = \bar{I}(0)\bar{\lambda}^0(t) + \displaystyle\int_0^t \bar{\lambda}(t-s)\bar{S}(s)\bar{\mathfrak{F}}(s)\,ds, \\ \bar{I}(t) = \bar{I}(0)F_0^c(t) + \displaystyle\int_0^t F^c(t-s)\bar{S}(s)\bar{\mathfrak{F}}(s)\,ds, \\ \bar{R}(t) = \bar{R}(0) + \bar{I}(0)F_0(t) + \displaystyle\int_0^t F(t-s)\bar{S}(s)\bar{\mathfrak{F}}(s)\,ds, \end{cases} \tag{2.1}$$

where $\bar{\lambda}^0(t) = \mathbb{E}[\lambda_{-1}(t)]$ and $\bar{\lambda}(t) = \mathbb{E}[\lambda_1(t)]$, F_0 (resp., F) is the distribution function of η_{-1} (resp., of η_1) and $F_0^c(t) = 1 - F_0(t)$, $F^c(t) = 1 - F(t)$. This convergence holds true provided that $\lambda \in D$ a.s. and, for some $\lambda^* > 0$, $0 \leq \lambda_j(t) \leq$

λ^* a.s. for all $j \in \mathbb{Z}$ and $t \geq 0$, see [7]. The original proof in [6] puts more restrictions on λ.

2.2. **The branching process approximation.** We fix a moderate number $M \ll N$ and wait until the time t_0 when the number of infected individuals in the population equals M (or is of the order of M), while the mean number $R_{\text{eff}} = \bar{S}(t_0) \int_0^\infty \bar{\lambda}(t)\, dt$ of individuals which an infected individual infects satisfies $R_{\text{eff}} < 1$. Then the epidemic is declining. It can be well approximated by the following non-Markovian continuous-time branching process. We will study its extinction time in the next section, and deduce a good approximation of the time we have to wait after t_0 for the epidemic to go extinct. Note that we approximate the proportion $\bar{S}(t)$ by $\bar{S}(t_0)$ for any $t \geq t_0$. This is quite reasonable, since between time 0 and the end of the epidemic, only a moderate number of individuals get infected, as the branching process approximation (and also upper bound) tells us, see below. Hence for large enough N, the proportion of susceptible individuals does not vary significantly.

Let us now describe in detail the branching process approximation of the final phase of the epidemic. We assume that there are M individuals who are infected at time t_0 and have been infected by someone who has recovered by time t_0. We arrange these M individuals in ascending order of their time of infection. This produces a sequence $(I_1, \tau_1), \dots, (I_M, \tau_M)$, where $\tau_1 < \tau_2 < \cdots < \tau_M < t_0$ are the times of infection.

We let $(X_1, \theta_1) = (I_1, \tau_1)$, and associate to X_1 a copy λ_1 of the random function λ. At any time $t > 0$, the individual X_1 gives birth (infects) at rate $\lambda_1(t - \theta_1)$, i.e., its descendants are born at the points of the point process

$$\int_{\theta_1}^{t \vee \theta_1} \int_0^\infty \mathbf{1}_{u \leq \lambda_1(s-\theta_1)} Q_1(ds, du),$$

where Q_1 is a standard Poisson random measure (abbreviated PRM) on $\mathbb{R}_+^2$. Let $\theta_{1,1} < \theta_{1,2} < \cdots$ be the points of this point process, and denote by $I_{1,1}, I_{1,2}, \dots$ the direct descendants of I_1. Note that there are finitely many of those (possibly none, in which case all the $\theta_{1,i}$ are chosen to be equal to $+\infty$), since our assumption $\int_0^\infty \bar{\lambda}(t)\, dt < \bar{S}(t_0)^{-1}$ implies that $\int_0^\infty \lambda(t)\, dt < \infty$ a.s.

We let

$$\theta_2 = \tau_2 \wedge \theta_{1,1} \quad \text{and} \quad X_2 = I_2 \mathbf{1}_{\tau_2 < \theta_{1,1}} + I_{1,1} \mathbf{1}_{\theta_{1,1} < \tau_2},$$

and we attach to X_2 a copy λ_2 of the random function λ, and the point process of the times of birth of the descendants of X_2 are those of the point process

$$\int_{\theta_2}^{t \vee \theta_2} \int_0^\infty \mathbf{1}_{u \leq \lambda_2(s-\theta_2)} Q_2(ds, du).$$

We let $\theta_{2,1} < \theta_{2,2} < \cdots$ be the points of this point process, and denote by $I_{2,1}, I_{2,2}, \dots$ the direct descendants of I_2. We next consider the collection $I_2, \dots, I_M$, $I_{1,2}, \dots, I_{2,1}, I_{2,2}, \dots$ in case we had $\theta_{1,1} < \tau_2$, and $I_3, \dots, I_M, I_{1,1}, I_{1,2}, \dots, I_{2,1}, \dots$ in case we had $\tau_2 < \theta_{1,1}$ (note that the two random variables differ a.s.), and we choose as X_3 the first born individual in this last collection.

We iterate, and define with this procedure a finite sequence. Indeed, since our branching process is subcritical, it goes extinct in finite time a.s. Note that the collection $\{(\lambda_i, Q_i) : i \geq 1\}$ is i.i.d., and for each $i \geq 1$, λ_i and Q_i are independent.

Now, inspired by the description done in [3, Part I, Section 1.2] of the branching process approximation of the initial phase of an epidemic, we couple this branching process with the epidemic, starting from time θ_1. For that, we first number from 1 to $N_0 = [S(t_0)N]$ all individuals in the population who were susceptible at time θ_1^-, and we let $\{U_i : i \geq 1\}$ be an i.i.d. sequence of $\mathcal{U}([0,1])$ random variables, which is independent of the branching process. For each $i \geq 1$, we now identify X_i with the individual $[U_i N_0] + 1$. It is easy to see that the branching process is a correct model of the succession of infections in the epidemic starting at time θ_1, as long as $[U_i N_0] + 1 \notin \{[U_1 N_0] + 1, \ldots, [U_{i-1} N_0] + 1\}$. Let $\breve{M}$ denote the random number of individuals in the above branching process. We note that the branching process (and in particular the number $\breve{M}$) does not depend upon the total population size N.

The probability that the branching process and the epidemic starting from time θ_1 coincide is the probability of the event

$$\cap_{i=2}^{\breve{M}} \{[U_i N_0] + 1 \notin \{[U_1 N_0] + 1, \ldots, [U_{i-1} N_0] + 1\}\,.$$

We first compute

$$\begin{aligned}
\mathbb{P}\Big(\cap_{i=2}^{\breve{M}} \{[U_i N_0] + 1 \notin \{[U_1 N_0] + 1, \ldots, [U_{i-1} N_0] + 1\} | \breve{M}\Big) & \\
&= \prod_{i=1}^{\breve{M}-1} \left(1 - \frac{i}{N_0}\right) \\
&= \exp\left\{ \sum_{i=1}^{\breve{M}-1} \log\left(1 - \frac{i}{N_0}\right) \right\} \\
&= \exp\left\{ - N_0^{-1} \sum_{i=1}^{\breve{M}-1} i + O(N_0^{-2}) \right\} \\
&= \exp\left\{ - \frac{\breve{M}(\breve{M}-1)}{2N_0} + O(N_0^{-2}) \right\} \\
&= 1 - \frac{\breve{M}(\breve{M}-1)}{2N_0} + O(N_0^{-2}).
\end{aligned}$$

It remains to take the expectation of this expression. Hence we need to compute the first two moments of $\breve{M}$. For that, we consider the branching process as a discrete-time Galton–Watson process. For $n \geq 1$, let Z_n denote the number of individuals of generation n, descendants of a unique ancestor at generation 0, where the number ξ of daughters of each individual follows the MixPoisson$\Big(\int_0^\infty \hat{\lambda}(t)\,dt\Big)$ distribution, that is, the conditional law of ξ, given that $\int_0^\infty \hat{\lambda}(t)\,dt = a$, is Poi$(a)$. It follows from well-known results on Galton–Watson processes (see, e.g., Athreya and

Ney [1, p. 4]) that $\mathbb{E}[Z_n] = R_{\text{eff}}^n$, $\text{Var}(Z_n) = \sigma^2 R_{\text{eff}}^{n-1} \frac{1-R_{\text{eff}}^n}{1-R_{\text{eff}}}$ and for $n \geq 1$, $\mathbb{E}[Z_n^2] = \sigma^2 R_{\text{eff}}^{n-1} \frac{1-R_{\text{eff}}^n}{1-R_{\text{eff}}} + R_{\text{eff}}^{2n}$ with $\sigma^2 = \text{Var}(\xi)$, while $\mathbb{E}[Z_0^2] = 1$. Our branching process is the sum of M independent copies of this Galton–Watson process. Consequently,

$$\begin{aligned}\mathbb{E}[\breve{M}] &= M\mathbb{E}\sum_{n=0}^{\infty} Z_n \\ &= M(1 + R_{\text{eff}} + R_{\text{eff}}^2 + \cdots) \\ &= \frac{M}{1 - R_{\text{eff}}}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[\breve{M}^2] &= M\mathbb{E}\left[\left(\sum_{n=0}^{\infty} Z_n\right)^2\right] + M(M-1)\left(\mathbb{E}\sum_{n=0}^{\infty} Z_n\right)^2 \\ &= M\mathbb{E}\sum_{n=0}^{\infty} Z_n^2 + 2M\mathbb{E}\sum_{n=0}^{\infty}\sum_{m>n} Z_n Z_m + \frac{M(M-1)}{(1-R_{\text{eff}})^2} \\ &= \frac{M}{(1-R_{\text{eff}})^2} + \frac{M\sigma^2}{(1-R_{\text{eff}})^3} + \frac{M(M-1)}{(1-R_{\text{eff}})^2} \\ &= \frac{M\sigma^2}{(1-R_{\text{eff}})^3} + \frac{M^2}{(1-R_{\text{eff}})^2}\,.\end{aligned}$$

An easy computation yields

$$\sigma^2 = \mathbb{E}\left[\left(\int_0^{\infty} \hat{\lambda}(t)\,dt\right)^2\right] + R_{\text{eff}}(1 - R_{\text{eff}})\,. \tag{2.2}$$

Putting together the above computations, we conclude the following statement.

Lemma 2.1. *The probability that, starting with M individuals infected at time t_0, whose parents in the epidemic genealogy have recovered at time t_0, while some of the M individuals may have infected others before time t_0, the epidemic evolves exactly as the above continuous-time branching process equals*

$$1 - \frac{M}{2N_0(1-R_{\text{eff}})}\left(\frac{M}{1-R_{\text{eff}}} + \frac{\sigma^2}{(1-R_{\text{eff}})^2} - 1\right) + O(N_0^{-2}),$$

where $N_0 = [S(t_0)N]$ and σ^2 is given by (2.2).

The above probability converges of course to 1 as $N_0 \to \infty$, and we know at which speed this occurs. Unfortunately, we do not know how the terms of order N_0^{-2} depend upon M. Note that, for large enough N_0, when the epidemic and the branching process do not coincide, we expect that their difference is minor. Moreover, the extinction time of the branching process is an upper bound of the extinction time of the epidemic.

3. The extinction time of the branching process associated to the varying infectivity model

For now on, $t \geq 0$ stands for $t - t_0$ ($t \geq t_0$). We define $\hat{\lambda}(t) := \bar{S}(t_0)\lambda(t)$. Let $Z(t)$ denote the number of descendants at time t of an individual born (i.e., infected) at time 0, in the continuous-time branching process which approximates the number of infected individuals at time t. This ancestor infects susceptible individuals during the time interval $[0, \eta]$, at the random and varying rate $\hat{\lambda}(t)$. His descendants have the same behavior, each one independently from all the others.

In this paper, we make the following assumption on the infectivity function:

Assumption (H) We shall assume that there exists a constant $\lambda^* > 0$ such that

$$\lambda(t) \leq \lambda^* \quad \text{almost surely for all } t \geq 0.$$

Let $T_{\text{ext}} = \inf\{t > 0 : Z(t) = 0\}$ denote the extinction time of the branching process, $G(s,t) = \mathbb{E}\left(s^{Z_t}\right)$, $|s| \leq 1$, denote the probability-generating function of $Z(t)$, and $F(t) = G(0,t)$ the distribution function of the extinction time.

3.1. Distribution function of the extinction time.

In this subsection, we will characterize the distribution function of the extinction time of Z as the unique solution of an integral equation. To this end, we imitate the computations done in the proof of Theorem 4.1 in [5]. We first start by determining the generating function $G(s,t)$ of Z in order next to deduce the distribution function of the extinction time.

Denote by $Z_0(t)$ the descendants of the ancestor at time t, and for $j \geq 1$, $Z_j(t)$ the descendants of the j-th direct descendant of the ancestor at time t after his/her birth. Then $\{Z_j(\cdot) : j \geq 0\}$ is a sequence of i.i.d. random processes which have the law of Z. In order to simplify our notations, we will write $\hat{\lambda}_0$ (resp., η_0) for the value of $\hat{\lambda}$ (resp., η) associated with Z_0. Formula (3.1) from [4] reads

$$Z_0(t) = \mathbb{1}_{\eta_0 > t} + \sum_{j=1}^{Q_0(t)} Z_j(t - t^j), \tag{3.1}$$

where $Q_0(t)$ is the number of direct descendants of the ancestor born on the time interval $(0,t]$. Moreover, $Q_0(t)$ is a counting process, which conditionally upon $\hat{\lambda}_0(\cdot)$, is a non-homogeneous Poisson process with varying intensity $\hat{\lambda}_0(t)$, and $0 < t^1 < t^2 < \cdots$ are the successive jump times of the process $Q_0(t)$.

We have the following result.

Proposition 3.1. *The probability-generating function G satisfies the integral equation*

$$G(s,t) = \mathbb{E}\left[s^{\mathbb{1}_{\eta > t}} \exp\left\{ \int_0^t \Big(G(s, t-u) - 1 \Big) \hat{\lambda}(u)\, du \right\} \right].$$

Proof. Since Z has the same law as Z_0, we first to compute $\mathbb{E}\left[s^{Z_0(t)}|\hat{\lambda}_0\right]$ in order to deduce the value of G. From (3.1), we deduce that

$$\begin{aligned}
\mathbb{E}\left[s^{Z_0(t)}|\hat{\lambda}_0\right] &= \sum_{k=0}^{\infty} s^{\mathbb{1}_{\eta_0>t}}\mathbb{P}(Q_0(t)=k|\hat{\lambda}_0)\mathbb{E}\left\{\prod_{j=1}^{k} s^{Z_j(t-t^j)}\Big|Q_0(t)=k,\hat{\lambda}_0\right\}\\
&= \sum_{k=0}^{\infty} s^{\mathbb{1}_{\eta_0>t}}\mathbb{P}(Q_0(t)=k|\hat{\lambda}_0)\mathbb{E}\left\{\prod_{j=1}^{k} G(s,t-t^j)\Big|Q_0(t)=k,\hat{\lambda}_0\right\}\\
&= \sum_{k=0}^{\infty} s^{\mathbb{1}_{\eta_0>t}}\mathbb{P}(Q_0(t)=k|\hat{\lambda}_0)\frac{k!}{\left(\int_0^t \hat{\lambda}_0(v)\,dv\right)^k}\\
&\quad\times\int_0^t\int_0^{u_k}\cdots\int_0^{u_2}\prod_{j=1}^{k} G(s,t-u_j)\hat{\lambda}_0(u_1)\ldots\hat{\lambda}_0(u_k)\,du_1\ldots du_k\\
&= s^{\mathbb{1}_{\eta_0>t}}\exp\left(-\int_0^t\hat{\lambda}_0(v)\,dv\right)\\
&\quad\times\sum_{k=0}^{\infty}\int_0^t\int_0^{u_k}\cdots\int_0^{u_2}\prod_{j=1}^{k} G(s,t-u_j)\hat{\lambda}_0(u_1)\ldots\hat{\lambda}_0(u_k)\,du_1\ldots du_k\\
&= s^{\mathbb{1}_{\eta_0>t}}\exp\left(-\int_0^t\hat{\lambda}_0(v)\,dv\right)\sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_0^t G(s,t-u)\hat{\lambda}_0(u)\,du\right)^k\\
&= s^{\mathbb{1}_{\eta_0>t}}\exp\left\{\int_0^t\Big(G(s,t-u)-1\Big)\hat{\lambda}_0(u)\,du\right\}.
\end{aligned}$$

The third equality exploits the well-known result on the law of the times of the jumps of a Poisson process on a given interval, given the number of those jumps (see Exercise 6.5.4 in [12], which treats the case of a constant rate, the general case follows via an obvious time change), and the fourth equality the conditional law of $Q_0(t)$, given $\hat{\lambda}_0$. We thus obtain

$$G(s,t)=\mathbb{E}\left[s^{\mathbb{1}_{\eta_0>t}}\exp\left\{\int_0^t\Big(G(s,t-u)-1\Big)\hat{\lambda}_0(u)\,du\right\}\right].$$

Since $(\hat{\lambda}_0,\eta_0)$ has the same law as $(\hat{\lambda},\eta)$, we can drop the subindices 0 in the last formula, yielding the formula of the statement. □

The term $s^{\mathbb{1}_{\eta>t}}$ can be written as follows: $s^{\mathbb{1}_{\eta>t}}=\mathbb{1}_{\eta\le t}+s\mathbb{1}_{\eta>t}$. From this, we deduce readily the following corollary for $F(t)=G(0,t)$.

Corollary 3.2. *The distribution function F of the extinction time of the branching process with one unique ancestor born at time 0 satisfies the integral equation*

$$F(t) = \mathbb{E}\left[\mathbb{1}_{\eta\le t}\exp\left\{\int_0^t \Big(F(t-u)-1\Big)\hat{\lambda}(u)\,du\right\}\right]. \tag{3.2}$$

The fact that (3.2) characterizes F follows from the next crucial result.

Proposition 3.3. *Equation* (3.2) *has a unique* $[0,1]$*-valued solution.*

Proof. The distribution function of the extinction time solves this equation. Let us show that this equation has at most one $[0,1]$-valued solution. To this end, suppose that the equation has two solutions F^1 and F^2 which are upper bounded by 1. We have

$$F^1(t)-F^2(t) = \mathbb{E}\Bigg[\mathbb{1}_{\eta\le t}\Bigg(\exp\left\{\int_0^t \Big(F^1(t-u)-1\Big)\hat{\lambda}(u)\,du\right\} - \exp\left\{\int_0^t \Big(F^2(t-u)-1\Big)\hat{\lambda}(u)\,du\right\}\Bigg)\Bigg].$$

From the fact that $|e^{-x}-e^{-y}|\le|x-y|$ for all $x,y>0$, we deduce that

$$\begin{aligned}\Big|F^1(t)-F^2(t)\Big| &\le \mathbb{E}\left[\int_0^t \hat{\lambda}(u)\Big|F^1(t-u)-F^2(t-u)\Big|\,du\right]\\ &\le \hat{\lambda}^*\int_0^t \Big|F^1(u)-F^2(u)\Big|\,du,\end{aligned}$$

where we have used Assumption (H) and the notation $\hat{\lambda}^* = \bar{S}(t_0)\lambda^*$. The desired result follows by combining this with Gronwall's lemma. □

3.2. **Epidemic with several individuals infected at the initial time.** Now we assume that several individuals have been infected before time 0 (or time t_0 in our epidemic model). We assume that M individuals are infected at time 0, whose parents in the epidemic genealogy have recovered by time 0, while some of the M individuals may have infected others before time 0. As a result, the exact number of infected individuals at time 0 is larger than M. However, since $R_{\text{eff}}<1$ and some of the M individuals may have been infected shortly before time 0 with probability close to 1, that number is less than $2M$, and in fact close to M. Note also that, while those M individuals have been infected before time 0, i.e., before t_0 in our original epidemic model, we expect that when those individuals start to infect susceptible individuals in the population, the proportion of susceptible individuals is well approximated by $\bar{S}(t_0)$. Since the dynamics of reproduction remains the same for all infected individuals resulting from each ancestor, from the branching property, we deduce the main result of this section.

Theorem 3.4. *The distribution function of the time we have to wait in order to see the extinction of the epidemic, if at time t_0 we have M infected individuals whose parents in the epidemic genealogy have recovered by time 0, is well approximated by*

$$H(t) = (F(t))^M .$$

4. Several examples of random function $\lambda(t)$

Our varying infectivity model is in fact a SIR/SEIR, in the sense that it allows an exposed period just after infection, during which $\lambda(t) = 0$. However, we do not introduce the E compartment (E for *exposed*, the status of an infected individual who is, just after being infected, in a latent period, not yet infectious), the I compartment including all infected individuals, whether latent or infectious. In all most used models, $\lambda(t)$ is piecewise constant, the jump times being random, following most classically an exponential distribution so that the stochastic model is Markovian and its law of large numbers limit is a system of ordinary differential equations (in contrast with the integral equation (2.1)).

We now review two classical examples of piecewise constant $\lambda(t)$, which correspond respectively to the SIR and the SEIR models and finally present the example of varying infectivity $\lambda(t)$ which we shall use in the next section for our comparison with the more classical SIR ODE model.

4.1. **The classical SIR model.** The simplest commonly used example of the infectivity $\lambda(t)$ is $\lambda(t) = \lambda \mathbb{1}_{t\le\eta}$, where λ is a positive constant and η is the random duration of the infectious period. In that case equation (3.2) takes the form

$$F(t) = \int_0^t \exp\left\{\lambda \int_0^r \Big(F(t-u) - 1\Big)\, du\right\} \mathbb{P}_\eta(dr).$$

In the particular case of a deterministic η (i.e., $\mathbb{P}_\eta = \delta_a$ with $a \in \mathbb{R}_+$), we have

$$F(t) = \mathbb{1}_{t\ge a} \exp\left\{\lambda \int_0^a \Big(F(t-u) - 1\Big)\, du\right\}$$

with $F(0) = 0$ and $F(a) = \exp(-\lambda a)$. The most commonly used model corresponds to η following an exponential distribution with parameter μ. In this case, the system of integral equations (2.1) simplifies as follows:

$$\begin{cases} \dfrac{dS(t)}{dt} = -\lambda \overline{S}(t) I(t), \\ \dfrac{dI(t)}{dt} = \big(\lambda \overline{S}(t) - \mu\big)\, I(t), \\ \dfrac{dR(t)}{dt} = \mu I(t). \end{cases}$$

If we linearize the second equation for $t \ge t_0$ by replacing $\overline{S}(t)$ by $\overline{S}(t_0)$, we obtain

$$I(t) = I(t_0) \exp\left[\big(\lambda \overline{S}(t_0) - \mu\big)\,(t - t_0)\right].$$

From this, it is easy to see that

$$\rho = \lambda \overline{S}(t_0) - \mu. \tag{4.1}$$

The fact that this formula is correct, although the deterministic model is not valid for $t \geq t_0$, is explained in [6]. Note also that solving equation (4.5) below gives the same result, as the reader can easily verify.

Let us now compute R_{eff}. An infected individual has infectious contacts at rate $\lambda \overline{S}(t_0)$. This means that the expected number of infectious contacts equals

$$R_{\text{eff}} = \lambda \overline{S}(t_0) \times \mathbb{E}[\eta] = \frac{\lambda \overline{S}(t_0)}{\mu}. \tag{4.2}$$

The approximating branching process is the continuous-time Markov branching process $(X(t))_{t\geq 0}$ which describes the number of descendants alive at time t of a unique ancestor born at time 0. Every individual in this population, independently of the others, lives for an exponential time with parameter μ, and during his/her lifetime he/she gives birth at rate $\lambda \overline{S}(t_0)$. His/her descendants reproduce according to the same procedure. We consider the subcritical case $\mu > \lambda \overline{S}(t_0)$. Let $G(s,t) = \mathbb{E}\left(s^{X(t)}\right)$, $|s| \leq 1$, be the probability-generating function of $X(t)$. On page 109 of Athreya and Ney [1], or in formula (5) of Iwasa, Nowak, and Michor [10], we find the explicit form

$$G(s,t) = \frac{\mu(s-1) - e^{-\rho t}(\lambda \overline{S}(t_0)s - \mu)}{\lambda \overline{S}(t_0)(s-1) - e^{-\rho t}(\lambda \overline{S}(t_0)s - \mu)},$$

with ρ as defined in (4.1). Let us define $T_{\text{ext}} = \inf\{t > 0 : X(t) = 0\}$. We notice that $F(t) = G(0,t) = \mathbb{P}(X_t = 0) = \mathbb{P}(T_{\text{ext}} \leq t)$ is the distribution function of the extinction time. From the expression for $G(s,t)$, we deduce the value of $F(t)$.

Proposition 4.1. *When starting with a single ancestor at time* 0*, the distribution function of the extinction time is given as*

$$F(t) = \frac{1 - e^{\rho t}}{1 - R_{\text{eff}} \times e^{\rho t}},$$

with R_{eff} *as defined in* (4.2).

4.2. **The classical SEIR model.** In this model, upon infection, an individual is first exposed (compartment E) for a period ξ, during which the individual is not infectious; then, he/she becomes infectious and remains so for a duration η, during which he/she infects susceptibles at a rate λ, and finally recovers. In that case, we have $\lambda(t) = \lambda \mathbf{1}_{\xi \leq t < \xi + \eta}$, and equation (3.2) takes the form

$$F(t) = \int_0^t \int_0^{t-r} \exp\left\{\lambda \int_s^{s+r} \Big(F(t-u) - 1\Big)\, du\right\} \mathbb{P}_{(\xi,\eta)}(ds, dr).$$

When ξ and η are deterministic, that is, $\mathbb{P}_{(\xi,\eta)}(ds,dr) = \delta_a(ds)\delta_b(dr)$ with $(a,b) \in \mathbb{R}_+^2$, we have

$$F(t) = \mathbb{1}_{t \geq a+b} \exp\left\{\lambda \int_a^{a+b} \Big(F(t-u) - 1\Big)\, du\right\} \quad \text{with } F(u) = 0 \text{ for all } u \in [0,a].$$

In case ξ and η are independent and follow exponential distributions with parameters γ and μ, respectively, the deterministic model obeys the ODE

$$\begin{cases} \dfrac{d\overline{S}(t)}{dt} = -\lambda\overline{S}(t)\overline{I}(t), \\ \dfrac{d\overline{E}(t)}{dt} = \lambda\overline{S}(t)\overline{I}(t) - \gamma\overline{E}(t), \\ \dfrac{d\overline{I}(t)}{dt} = \gamma\overline{E}(t) - \mu\overline{I}(t), \\ \dfrac{d\overline{R}(t)}{dt} = \mu\overline{I}(t). \end{cases}$$

In this model, again $R_{\text{eff}} = \frac{\lambda\overline{S}(t_0)}{\mu}$. Solving equation (4.5) below for ρ, we find

$$\rho = \frac{1}{2}\left[\sqrt{(\gamma-\mu)^2 + 4\gamma\overline{S}(t_0)\lambda} - (\mu+\gamma)\right].$$

4.3. **Our varying infectivity model.** We again define $\hat{\lambda}(t) = \overline{S}(t_0)\lambda(t)$. The infectivity $\hat{\lambda}(t)$ is first zero (corresponding to the latency period) followed by a gradual increase for some days, and then $\hat{\lambda}(t)$ starts decreasing down towards zero, hitting it when the individual has recovered (see Figure 1).

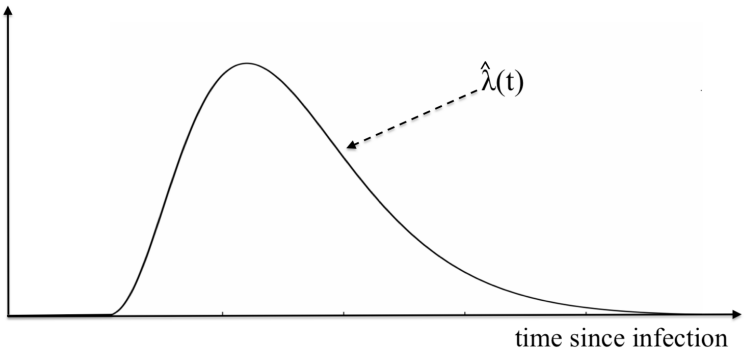

FIGURE 1. Example of trajectory of $\hat{\lambda}(t)$.

In the computations of Section 5 below, we use a piecewise linear $\hat{\lambda}(t)$, which allows the function to depend upon a small number of parameters (see Figure 2).

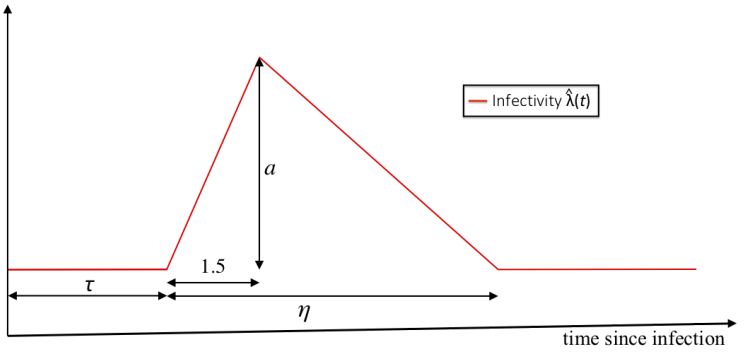

FIGURE 2. Trajectory of $\hat{\lambda}(t)$ used for the comparisons below.

Here τ is the duration of the exposed period, η that of the infectious period. We have arbitrarily fixed the length of the period of increase to 1.5 days, and taken the maximum value a to be a deterministic quantity at our disposal. In other words, in this case we have

$$\hat{\lambda}(t) = \begin{cases} 0 & \text{if } t < \tau, \\ \dfrac{a}{1.5}\,(t-\tau) & \text{if } \tau \le t < \tau + 1.5, \\ a\dfrac{\tau+\eta-t}{\eta-1.5} & \text{if } \tau + 1.5 \le t < \tau + \eta, \\ 0 & \text{if } \tau + \eta < t. \end{cases} \tag{4.3}$$

Let $\mathcal{J}$ be the joint law of τ and η. From Corollary 3.2, we deduce that

$$F(t) = \mathbb{E}\left[\mathbb{1}_{\zeta \le t} \exp\left\{ \frac{a}{1.5} \int_\tau^{\tau+1.5} (F(t-u)-1)(u-\tau)\,du \right.\right. \\ \left.\left. + \frac{a}{\eta-1.5} \int_{\tau+1.5}^{\tau+\eta} (F(t-u)-1)(\tau+\eta-u)\,du \right\}\right]$$

with $\zeta = \tau + \eta$. Thus, we obtain

$$F(t) = \int_0^t \int_0^t \mathbb{1}_{s+r \le t} \exp\left\{ \frac{a}{1.5} \int_s^{s+1.5} (F(t-u)-1)(u-s)\,du \right. \\ \left. + \frac{a}{r-1.5} \int_{s+1.5}^{s+r} (F(t-u)-1)(s+r-u)\,du \right\} \mathcal{J}(ds, dr).$$

The effective reproduction number is defined by

$$R_{\text{eff}} = \mathbb{E}\left[\int_0^\infty \hat{\lambda}(t)\,dt\right], \tag{4.4}$$

and the rate of decrease ρ of the number of infected individuals is the unique solution of

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\hat{\lambda}(t)\,dt\right] = 1 \tag{4.5}$$

(see [6, Theorem 2.3]).

5. Comparison between our varying infectivity model and a Markovian SIR model

In this section, we compare the distribution function of the extinction time in our varying infectivity model with that of a Markovian SIR model with the same R_{eff}, which is the effective reproduction number at time t_0, and the same rate of decrease ρ of the number of infected individuals. The law of large numbers limit of such a Markovian SIR model is a system of ODEs. The approximating branching process is a continuous-time Markov branching process.

In the following, we assume that the random variables τ and η defined in (4.3) are independent, $\tau \sim \mathcal{U}(1.5, 2.5)$ and $\eta \sim \mathcal{U}(7, 13)$. These values are somewhat arbitrary. They are compatible with the results in [9] concerning the COVID epidemic.

5.1. **Approximation of the distribution function of the extinction time in the varying infectivity model.** Since it is not possible to obtain an explicit solution of (3.2), then we will use the approximation made in Appendix A. In other words, we will consider the following approximate solution (whose convergence is established in Appendix A below):

$$F_n\left(\frac{k}{n}\right) = \mathbb{E}\left[\mathbb{1}_{\tau+\eta\le\frac{k}{n}}\exp\left\{\sum_{\ell=1}^{k}\left(F_n\left(\frac{k-\ell}{n}\right)-1\right)\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}\hat{\lambda}(u)\,du\right\}\right].$$

Let us define $\xi_{n,\ell} = \displaystyle\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}\hat{\lambda}(u)\,du$. It is easy to see that $\xi_{n,\ell} \approx \frac{\hat{\lambda}\left(\frac{\ell}{n}\right)}{n}$. Combining this with (4.3), we deduce that

$$\xi_{n,\ell} \approx \frac{a}{1.5}\left(\frac{\ell}{n}-\tau\right)\mathbb{1}_{\tau\le\frac{\ell}{n}<\tau+1.5} + a\left(\frac{\tau+\eta-\frac{\ell}{n}}{\eta-1.5}\right)\mathbb{1}_{\tau+1.5\le\frac{\ell}{n}<\tau+\eta}.$$

Now, using the fact that the random variables τ and η are independent, $\tau \sim \mathcal{U}(1.5, 2.5)$ and $\eta \sim \mathcal{U}(7, 13)$, we deduce that

$$F_n\left(\frac{k}{n}\right) \approx \frac{1}{6}\int_{1.5}^{2.5}\int_{7}^{13}\mathbb{1}_{x+y\le\frac{k}{n}}\exp\left\{\sum_{\ell=1}^{k}\left(F_n\left(\frac{k-\ell}{n}\right)-1\right)\frac{a}{1.5}\left(\frac{\ell}{n}-x\right)\mathbb{1}_{x\le\frac{\ell}{n}<x+1.5}\right\}$$
$$\times\exp\left\{\sum_{\ell=1}^{k}\left(F_n\left(\frac{k-\ell}{n}\right)-1\right)a\frac{x+y-\frac{\ell}{n}}{y-1.5}\mathbb{1}_{x+1.5\le\frac{\ell}{n}<x+y}\right\}dxdy$$

$$\approx \frac{1}{6}\frac{1}{n^2}\sum_{j=7n}^{13n}\sum_{i=1.5n}^{2.5n}\mathbb{1}_{i+j\le k}\exp\left\{\sum_{\ell=i}^{i+1.5n}\left(F_n\left(\frac{k-\ell}{n}\right)-1\right)(\ell-i)\,\frac{a}{1.5n^2}\right\}$$
$$\times\exp\left\{\sum_{\ell=i+1.5n}^{i+j}\left(F_n\left(\frac{k-\ell}{n}\right)-1\right)\frac{i+j-\ell}{j-1.5n}\frac{a}{n}\right\}. \tag{5.1}$$

5.2. **Computation of R_{eff}.** Recall (4.4). We first compute the random quantity $\int_0^\infty \hat{\lambda}(t)\,dt$. This is the surface below the curve $\hat{\lambda}(t)$, i.e., the surface of the union of two triangles, and $\int_0^\infty \hat{\lambda}(t)\,dt = \dfrac{a\eta}{2}$. Therefore, we have

$$R_{\text{eff}} = \frac{a}{2}\mathbb{E}[\eta] = \frac{a}{2}\times 10 = 5a.$$

5.3. **Resolution of equation** (4.5)**.** From (4.3), we have

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\hat{\lambda}(t)\,dt\right] = a\,(A_\rho + B_\rho)$$

with

$$A_\rho = \mathbb{E}\left(\int_\tau^{\tau+1.5} e^{-\rho t}\frac{t-\tau}{1.5}\,dt\right) \quad\text{and}\quad B_\rho = \mathbb{E}\left(\int_{\tau+1.5}^{\tau+\eta} e^{-\rho t}\frac{\tau+\eta-t}{\eta-1.5}\,dt\right).$$

Using the fact that τ and η are independent, $\tau\sim\mathcal{U}(1.5,2.5)$, $\eta\sim\mathcal{U}(7,13)$, it is easy to check that

$$A_\rho = \frac{1}{\rho}\left(e^{-1.5\rho}-e^{-2.5\rho}\right)\left[\frac{1}{1.5\rho^2}-e^{-1.5\rho}\left(\frac{1}{\rho}+\frac{1}{1.5\rho^2}\right)\right]$$

and

$$B_\rho = \frac{1}{\rho}\left(e^{-1.5\rho}-e^{-2.5\rho}\right)\left\{e^{-1.5\rho}\left(\frac{1}{\rho}-\frac{1}{6\rho^2}\log\left(\frac{11.5}{5.5}\right)\right)+\frac{1}{\rho^2}\mathbb{E}\left[\frac{e^{-\rho\eta}}{(\eta-1.5)}\right]\right\}.$$

Note that the mapping $\rho\mapsto\mathbb{E}\int_0^\infty e^{-\rho t}\hat{\lambda}(t)\,dt$ is decreasing. Consequently, it is easy to compute an approximate solution of equation (4.5).

5.4. **Comparison of the distributions and the expectations of the extinction time between our varying infectivity model and a Markovian SIR model.** In what follows, we compare the extinction time in our varying infectivity model and in the Markovian SIR model with the same R_{eff} and ρ. Note that we compare F's and not H's (see the notations in Section 3). Of course the relevant quantities are rather the H's. It is easy to deduce from our results the corresponding comparison of the H's for various values of M. We compare the distribution of the extinction time of our varying infectivity model given in (5.1) and of the extinction time of the Markovian SIR model given in Proposition 4.1 (see Figures 3 and 4).

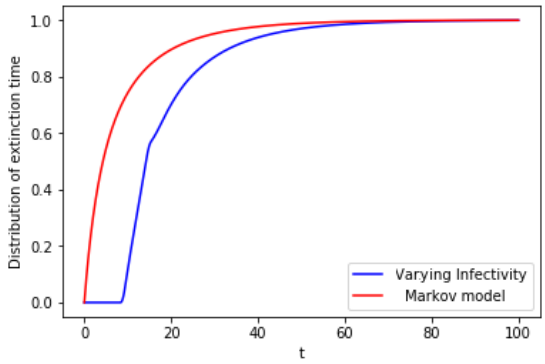

FIGURE 3. Comparison of models with the same $R_{\text{eff}} = 0.66$ and $\rho = -0.0683$.

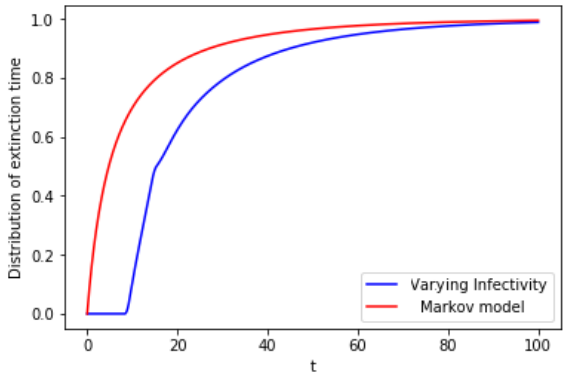

FIGURE 4. Comparison of models with the same $R_{\text{eff}} = 0.8$ and $\rho = -0.03816$.

We also compare the expectations of the extinction times of our varying infectivity model and of a Markovian SIR model. To this end, recall that the extinction time can be rewritten in the form $T_{\text{ext}} = \inf\{t - t_0 : I(t - t_0) = 0\}$. Thus, for the ODE SIR model, we obtain

$$\mathbb{E}[T_{\text{ext}}] = \int_0^\infty \mathbb{P}(T_{\text{ext}} > t)\, dt = \int_0^\infty (1 - F(t))\, dt = \frac{(1 - R_{\text{eff}})}{\rho R_{\text{eff}}} \ln(1 - R_{\text{eff}}),$$

where we have used the formula of Proposition 4.1 for $F(t)$.

For the varying infectivity model, we obtain

$$\mathbb{E}[T_{\text{ext}}] = \int_0^\infty \mathbb{P}(T_{\text{ext}} > t)\, dt = \int_0^\infty (1 - F_n(t))\, dt \approx \frac{1}{n} \sum_{k=1}^{n\Lambda} \left(1 - F_n\left(\frac{k}{n}\right)\right),$$

where Λ is the point where we stop the calculation of the integral of $1 - F_n(t)$. The next table displays the comparison of those expectations.

	$R_{\text{eff}} = 0.66$ $\rho = -0.0683$	$R_{\text{eff}} = 0.8$ $\rho = -0.03816$
Varying infectivity model	$\mathbb{E}[T_{\text{ext}}] \approx 18.7854$	$\mathbb{E}[T_{\text{ext}}] \approx 22.6568$
Markov SIR model	$\mathbb{E}[T_{\text{ext}}] = 8.1369$	$\mathbb{E}[T_{\text{ext}}] = 10.544$

6. Conclusion

Our comparison shows that in the final phase of the epidemic, the varying infectivity SIR model (in fact, its branching process approximation) tends to take more time to extinct than the branching process approximation of the Markovian SIR model. This is not too surprising, since the varying infectivity model has a memory, contrary to the Markovian and ODE SIR models. This fact is easily seen when there is a sudden change in the propagation of the epidemic, like the beginning of the lockdown that several countries established during the recent COVID epidemic. The authors who use an ODE model change the infection rate gradually, starting with the beginning of the lockdown, while in reality the change of the infection rate was very sudden. This is a way to compensate the lack of memory of ODE models. We believe that the fact that the varying infectivity SIR model takes more time than the ODE SIR model to forget its past explains why it takes more time to go extinct. The varying infectivity SIR model is more complex than the more classical ODE SIR model, and this probably explains why most authors who quote the seminal 1927 paper of Kermack and McKendrick [11] refer only to the very particular case of constant coefficients, studied in section 3.2 of that paper. Of course, it is very tempting and sometimes preferable to use simple models, which allow to draw more conclusions. However, it is crucial to understand the biases a simplified model introduces, compared to more realistic models. In this paper, we have identified one of those biases, namely the shortening of the final phase of the epidemic. In future work, we intend to do similar computations with various classes of varying infectivity models in order to confirm these first conclusions.

Appendix A. Approximation of the distribution function of the extinction time

We define a sequence of functions $\{F_n : n \geq 1\}$ which will allow us to approach the solution of equation (3.2). To this end, for each $k \in \mathbb{Z}_+$, we set

$$F_n\left(\frac{k}{n}\right) = \mathbb{E}\left[\mathbb{1}_{\eta \leq \frac{k}{n}} \exp\left\{\sum_{\ell=1}^{k}\left(F_n\left(\frac{k-\ell}{n}\right) - 1\right)\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}} \lambda(u)\,du\right\}\right], \tag{A.1}$$

and for each $t \in \left[\frac{k}{n}, \frac{k+1}{n}\right)$,

$$F_n(t) = \mathbb{E}\left[\mathbb{1}_{\eta \le t} \exp\left\{\sum_{\ell=1}^{k-1}\left(F_n\left(\frac{k-\ell}{n}\right)-1\right)\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}} \lambda(u)\,du - \int_{\frac{k-1}{n}}^{t}\lambda(u)\,du\right\}\right]. \tag{A.2}$$

The goal of this appendix is to prove that as $n \to +\infty$, $\{F_n(t) : t > 0\} \to \{F(t) : t > 0\}$ in $D([0,+\infty))$, where F is the unique solution of (3.2).

We first check the following.

Lemma A.1. *For any $k \in \mathbb{Z}_+$, we have*

$$F_n\left(\frac{k}{n}\right) \le F\left(\frac{k}{n}\right) \le 1.$$

Proof. Let $k \in \mathbb{Z}_+$. We first note that $F(t) \le 1$ (since F is a distribution function). To prove the next assertion, we will proceed by recurrence on k. It is clear that $F_n(0) = 0$. Let us now suppose that $F_n\left(\frac{\ell}{n}\right) \le F\left(\frac{\ell}{n}\right)$ for all $1 \le \ell \le k-1$. Now let us show that $F_n\left(\frac{k}{n}\right) \le F\left(\frac{k}{n}\right)$. We have

$$\begin{aligned}
F\left(\frac{k}{n}\right) &= \mathbb{E}\left[\mathbb{1}_{\eta \le \frac{k}{n}} \exp\left\{\int_0^{\frac{k}{n}}\left(F\left(\frac{k}{n}-u\right)-1\right)\lambda(u)\,du\right\}\right] \\
&\ge \mathbb{E}\left[\mathbb{1}_{\eta \le \frac{k}{n}} \exp\left\{\sum_{\ell=1}^{k}\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}\left(F\left(\frac{k-\ell}{n}\right)-1\right)\lambda(u)\,du\right\}\right] \\
&\ge \mathbb{E}\left[\mathbb{1}_{\eta \le \frac{k}{n}} \exp\left\{\sum_{\ell=1}^{k}\left(F_n\left(\frac{k-\ell}{n}\right)-1\right)\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}\lambda(u)\,du\right\}\right] \\
&= F_n\left(\frac{k}{n}\right),
\end{aligned}$$

where we have used the fact that F is non-decreasing and the recurrence assumption. □

The previous result extends to all t.

Lemma A.2. *For any $t \ge 0$, we have*

$$F_n(t) \le F(t) \le 1.$$

Proof. We first note that

$$\int_0^t (1-F(t-u))\,\lambda(u)\,du \le \sum_{\ell=1}^{[nt]}\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}(1-F(t-u))\,\lambda(u)\,du + \int_{\frac{[nt]}{n}}^{t}\lambda(u)\,du.$$

From the fact that F is non-decreasing and $\frac{[nt]}{n} \le t$, we deduce that

$$\int_0^t (1-F(t-u))\,du \le \sum_{\ell=1}^{[nt]}\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}\left(1-F\left(\frac{[nt]-\ell}{n}\right)\right)\lambda(u)\,du + \int_{\frac{[nt]}{n}}^{t}\lambda(u)\,du,$$

$$\int_0^t (F(t-u)-1)\,du \geq \sum_{\ell=1}^{[nt]} \left(F_n\left(\frac{\lceil nt\rceil - \ell}{n}\right) - 1\right) \int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}} \lambda(u)\,du - \int_{\frac{[nt]}{n}}^{t} \lambda(u)\,du,$$

where we have used Lemma A.1 for the last inequality. The desired result follows by combining the last inequality with (A.2). □

We have the following result.

Proposition A.3. *Let $T > 0$. Then there exists a constant C such that, for all $n \geq 1$ and $0 < s < t < T$,*

$$-\frac{C}{n} - C(t-s) \leq F_n(t) - F_n(s) \leq C(t-s) + \phi(t) - \phi(s) + \frac{C}{n},$$

where $\phi(t) = \mathbb{P}(\eta \leq t)$ is the distribution function of η.

For the proof of this proposition, we will need several technical lemmas. In order to simplify the notations below we let

$$\begin{aligned} a_n(k) &= \left[F_n\left(\frac{k+1}{n}\right) - F_n\left(\frac{k}{n}\right)\right]^-, \\ b_n(k) &= \left[F_n\left(\frac{k+1}{n}\right) - F_n\left(\frac{k}{n}\right)\right]^+. \end{aligned} \tag{A.3}$$

Let us define, for all $n \geq 1$, $k \in \mathbb{Z}_+$,

$$\Lambda_n(k) = \sum_{\ell=1}^{k} \left(F_n\left(\frac{k-\ell}{n}\right) - 1\right) \int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}} \lambda(u)\,du \leq 0 \tag{A.4}$$

(see Lemma A.1), and let us rewrite (A.1) in the form

$$F_n\left(\frac{k}{n}\right) = \mathbb{E}\left[\mathbb{1}_{\eta \leq \frac{k}{n}} \exp(\Lambda_n(k))\right]. \tag{A.5}$$

Lemma A.4. *For any $n \geq 1$, $k \in \mathbb{Z}_+$, we have*

$$A_1(n,k) \leq F_n\left(\frac{k+1}{n}\right) - F_n\left(\frac{k}{n}\right) \leq A_2(n,k)$$

with

$$A_1(n,k) = \exp\left\{ -\frac{\lambda^*}{n}\left[\sum_{\ell=0}^{k-1} a_n(\ell) + 1\right]\right\} - 1 \tag{A.6}$$

and

$$A_2(n,k) = \exp\left\{ \frac{\lambda^*}{n} \sum_{\ell=0}^{k-1} b_n(\ell)\right\} - 1 + \mathbb{P}\left(\frac{k}{n} < \eta \leq \frac{k+1}{n}\right). \tag{A.7}$$

Proof. Recalling (A.4) and (A.5), we first note that

$$\Lambda_n(k+1)-\Lambda_n(k)=\sum_{\ell=1}^{k}\left(F_n\left(\frac{k+1-\ell}{n}\right)-F_n\left(\frac{k-\ell}{n}\right)\right)\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}\lambda(u)\,du - \int_{\frac{k}{n}}^{\frac{k+1}{n}}\lambda(u)\,du.$$

It follows that

$$-\Big(\Lambda_n(k+1)-\Lambda_n(k)\Big)^{-}\geq -\sum_{\ell=1}^{k}\left(F_n\left(\frac{k+1-\ell}{n}\right)-F_n\left(\frac{k-\ell}{n}\right)\right)^{-}\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}\lambda(u)\,du - \int_{\frac{k}{n}}^{\frac{k+1}{n}}\lambda(u)\,du$$

and

$$\Big(\Lambda_n(k+1)-\Lambda_n(k)\Big)^{+}\leq \sum_{\ell=1}^{k}\left(F_n\left(\frac{k+1-\ell}{n}\right)-F_n\left(\frac{k-\ell}{n}\right)\right)^{+}\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}\lambda(u)\,du.$$

Thus, we have

$$\begin{aligned}
F_n\left(\frac{k+1}{n}\right)-F_n\left(\frac{k}{n}\right)&=\mathbb{E}\left[\mathbb{1}_{\eta\leq\frac{k+1}{n}}\exp(\Lambda_n(k+1))-\mathbb{1}_{\eta\leq\frac{k}{n}}\exp(\Lambda_n(k))\right]\\
&=\mathbb{E}\Big[\left(\mathbb{1}_{\eta\leq\frac{k+1}{n}}-\mathbb{1}_{\eta\leq\frac{k}{n}}\right)\exp(\Lambda_n(k+1))\\
&\qquad+\mathbb{1}_{\eta\leq\frac{k}{n}}\left(\exp(\Lambda_n(k+1))-\exp(\Lambda_n(k))\right)\Big]\\
&\leq\mathbb{E}\Big[\mathbb{1}_{\eta\leq\frac{k+1}{n}}-\mathbb{1}_{\eta\leq\frac{k}{n}}\\
&\qquad+\mathbb{1}_{\eta\leq\frac{k}{n}}\exp(\Lambda_n(k))\left(\exp(\Lambda_n(k+1)-\Lambda_n(k))-1\right)\Big]\\
&\leq\mathbb{P}\left(\frac{k}{n}<\eta\leq\frac{k+1}{n}\right)+\mathbb{E}\left(\exp\left[(\Lambda_n(k+1)-\Lambda_n(k))^{+}\right]-1\right)\\
&\leq\mathbb{E}\left[\exp\left\{\sum_{\ell=1}^{k}\left(F_n\left(\frac{k+1-\ell}{n}\right)-F_n\left(\frac{k-\ell}{n}\right)\right)^{+}\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}}\lambda(u)\,du\right\}\right]\\
&\quad-1+\mathbb{P}\left(\frac{k}{n}<\eta\leq\frac{k+1}{n}\right)\\
&\leq A_2(n,k),
\end{aligned}$$

where we have used (A.3) and (A.7) in the last inequality. We also have

$$\begin{aligned}
F_n\left(\frac{k+1}{n}\right)-F_n\left(\frac{k}{n}\right)&=\mathbb{E}\Big[\mathbb{1}_{\eta\leq\frac{k+1}{n}}\exp(\Lambda_n(k+1))-\mathbb{1}_{\eta\leq\frac{k}{n}}\exp(\Lambda_n(k))\Big]\\
&\geq\mathbb{E}\Big[\mathbb{1}_{\eta\leq\frac{k}{n}}\exp(\Lambda_n(k))\big(\exp(\Lambda_n(k+1)-\Lambda_n(k))-1\big)\Big]\\
&\geq\mathbb{E}\Big[\mathbb{1}_{\eta\leq\frac{k}{n}}\exp(\Lambda_n(k))\big(\exp(-(\Lambda_n(k+1)-\Lambda_n(k))^{-})-1\big)\Big]\\
&\geq\mathbb{E}\Big[\exp(-(\Lambda_n(k+1)-\Lambda_n(k))^{-})-1\Big].
\end{aligned}$$

Combining the above arguments with (A.3) and (A.6), we deduce that

$$F_n\left(\frac{k+1}{n}\right) - F_n\left(\frac{k}{n}\right) \geq A_1(n,k). \qquad \square$$

Recall (A.3). We have the following lemma.

Lemma A.5. *Let $T > 0$. Then there exists a constant C such that, for all $n \geq 1$ and $0 \leq \frac{k}{n} < T$,*

$$\sum_{\ell=0}^{k} a_n(\ell) \leq C \quad \text{and} \quad \sum_{\ell=0}^{k} b_n(\ell) \leq C.$$

Proof. Let us show the first assertion. For this, we first prove that

$$a_n(k) \leq r\,(1+r)^{k-1} \quad \text{with } r = \frac{\lambda^*}{n}.$$

According to Lemma A.4, we have

$$\begin{aligned} a_n(k) \leq -A_1(n,k) &= 1 - \exp\left\{ -r\left[\sum_{\ell=0}^{k-1} a_n(\ell) + 1\right]\right\} \\ &\leq r\left(\sum_{\ell=0}^{k-1} a_n(\ell) + 1\right). \end{aligned}$$

However, it is not hard to see that $a_n(0) = 0$ and $a_n(1) \leq r$. Let us suppose $a_n(\ell) \leq r\,(1+r)^{\ell-1}$ for all $1 \leq \ell \leq k-1$. Thus, it is easy to see that

$$\begin{aligned} a_n(k) &\leq r\left(1 + r + r(1+r) + \ldots + r(r+1)^{k-2}\right) \\ &= r\left(1 + r\sum_{i=1}^{k-1}(1+r)^{i-1}\right) = r(1+r)^{k-1}. \end{aligned}$$

Consequently, since $\frac{k}{n} \leq T$, we have

$$\sum_{\ell=0}^{k} a_n(\ell) = \sum_{\ell=1}^{k} a_n(\ell) \leq \sum_{\ell=1}^{k} r(1+r)^{\ell-1} = (1+r)^k - 1 \leq e^{rk} \leq e^{\lambda^* T} \leq C_T.$$

We now show the second assertion. We first have $b_n(0) = F_n\left(\frac{1}{n}\right)$. Then we have

$$\begin{aligned} \sum_{\ell=0}^{k} b_n(\ell) &= F_n\left(\frac{1}{n}\right) + \sum_{\ell=1}^{k}\left(F_n\left(\frac{\ell+1}{n}\right) - F_n\left(\frac{\ell}{n}\right)\right) + \sum_{\ell=1}^{k} a_n(\ell) \\ &= F_n\left(\frac{k+1}{n}\right) + \sum_{\ell=1}^{k} a_n(\ell) \\ &\leq 1 + \sum_{\ell=1}^{k} a_n(\ell), \end{aligned}$$

where we have used Lemma A.1 in the last inequality. The desired result follows by combining this with the first assertion. $\square$

Lemma A.6. *Let $T > 0$. Then there exists a constant C such that, for all $n \geq 1$ and $0 \leq \frac{\ell}{n} < \frac{k}{n} < T$,*

$$-C\left(\frac{k-\ell}{n}\right) \leq F_n\left(\frac{k}{n}\right) - F_n\left(\frac{\ell}{n}\right) \leq C\left(\frac{k-\ell}{n}\right) + \phi\left(\frac{k}{n}\right) - \phi\left(\frac{\ell}{n}\right),$$

where $\phi(t) = \mathbb{P}(\eta \leq t)$ is the distribution function of the random variable η.

Proof. Recall (A.6) and (A.7). We have

$$\begin{aligned} -A_1(n,k) &= 1 - \exp\left\{ -\frac{\lambda^*}{n}\left[\sum_{\ell=0}^{k-1} a_n(\ell) + 1\right]\right\} \\ &\leq \frac{\lambda^*}{n}\left[\sum_{\ell=0}^{k-1} a_n(\ell) + 1\right] \\ &\leq \frac{C}{n}, \end{aligned}$$

where we have used Lemma A.5. However, we have

$$\begin{aligned} A_2(n,k) &= \exp\left\{\frac{\lambda^*}{n}\sum_{\ell=0}^{k-1} b_n(\ell)\right\} - 1 + \mathbb{P}\Big(\frac{k}{n} < \eta \leq \frac{k+1}{n}\Big) \\ &\leq C\frac{\lambda^*}{n}\exp\left\{C\frac{\lambda^*}{n}\right\} + \mathbb{P}\Big(\frac{k}{n} < \eta \leq \frac{k+1}{n}\Big) \\ &\leq \frac{C}{n} + \mathbb{P}\Big(\frac{k}{n} < \eta \leq \frac{k+1}{n}\Big), \end{aligned}$$

where we have used the fact that $e^x - 1 \leq xe^x$ for all $x \geq 0$ and Lemma A.5. Now combining the above arguments with Lemma A.4, we deduce that

$$-\frac{C}{n} \leq F_n\left(\frac{k+1}{n}\right) - F_n\left(\frac{k}{n}\right) \leq \frac{C}{n} + \mathbb{P}\Big(\frac{k}{n} < \eta \leq \frac{k+1}{n}\Big).$$

However, we note that

$$F_n\left(\frac{k}{n}\right) - F_n\left(\frac{\ell}{n}\right) = \sum_{j=\ell}^{k-1}\left(F_n\Big(\frac{j+1}{n}\Big) - F_n\Big(\frac{j}{n}\Big)\right).$$

The desired result follows by combining this with the previous inequalities. □

Let us define, for all $n \geq 1$, $t > 0$ with $k = \lceil nt \rceil$,

$$\Lambda_n(t) = \sum_{\ell=1}^{k-1}\left(F_n\left(\frac{k-\ell}{n}\right) - 1\right)\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}} \lambda(u)\,du - \int_{\frac{k-1}{n}}^{t} \lambda(u)\,du \tag{A.8}$$

(see Lemma A.1) and let us rewrite (A.2) in the form

$$F_n(t) = \mathbb{E}\Big[\mathbb{1}_{\eta \leq t}\exp(\Lambda_n(t))\Big]. \tag{A.9}$$

Lemma A.7. *Let $T > 0$. Then there exists a constant C such that, for all $n \geq 1$ and $0 < \frac{\ell-1}{n} < s < \frac{\ell}{n} < \frac{k}{n} < t < \frac{k+1}{n} < T$,*

$$(\Lambda_n(t) - \Lambda_n(k))^+ = 0, \quad (\Lambda_n(t) - \Lambda_n(k))^- \leq C\left(t - \frac{k}{n}\right),$$
$$(\Lambda_n(\ell) - \Lambda_n(s))^+ \leq \frac{C}{n} \quad \textit{and} \quad (\Lambda_n(\ell) - \Lambda_n(s))^- \leq \frac{C}{n} + C\left(\frac{\ell}{n} - s\right),$$

where $\Lambda_n(\cdot)$ is as defined in (A.4).

Proof. From (A.4) and (A.8), we have

$$-\lambda^*\left(t - \frac{k}{n}\right) \leq \Lambda_n(t) - \Lambda_n(k) = -\int_{\frac{k}{n}}^{t} \lambda(u)\, du \leq 0.$$

Thus, we obtain the first two assertions. In the same way, from (A.4) and (A.8) we have

$$\begin{aligned}\Lambda_n(\ell) - \Lambda_n(s) &= \sum_{j=1}^{\ell-2}\left(F_n\left(\frac{\ell-j}{n}\right) - F_n\left(\frac{\ell-1-j}{n}\right)\right)\int_{\frac{j-1}{n}}^{\frac{j}{n}} \lambda(u)\, du - \int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}} \lambda(u)\, du \\ &\quad + \left(F_n\left(\frac{1}{n}\right) - 1\right)\int_{\frac{\ell-2}{n}}^{\frac{\ell-1}{n}} \lambda(u)\, du + \int_{\frac{\ell-2}{n}}^{s} \lambda(u)\, du \\ &= \sum_{j=1}^{\ell-2}\left(F_n\left(\frac{\ell-j}{n}\right) - F_n\left(\frac{\ell-1-j}{n}\right)\right)\int_{\frac{j-1}{n}}^{\frac{j}{n}} \lambda(u)\, du \\ &\quad + F_n\left(\frac{1}{n}\right)\int_{\frac{\ell-2}{n}}^{\frac{\ell-1}{n}} \lambda(u)\, du - \int_{s}^{\frac{\ell}{n}} \lambda(u)\, du.\end{aligned}$$

Combining this with Lemmas A.1, A.5 and (A.3), we deduce that

$$(\Lambda_n(\ell) - \Lambda_n(\ell-1, s))^+ \leq \frac{C}{n} \quad \text{and} \quad (\Lambda_n(\ell) - \Lambda_n(\ell-1, s))^- \leq \frac{C}{n} + C\left(\frac{\ell}{n} - s\right).$$

□

Lemma A.8. *Let $T > 0$. Then there exists a constant C such that, for all $n \geq 1$ and $0 < \frac{\ell-1}{n} < s < \frac{\ell}{n} < \frac{k}{n} < t < \frac{k+1}{n} < T$,*

$$-\frac{C}{n} - C\left(\frac{\ell}{n} - s\right) \leq F_n\left(\frac{\ell}{n}\right) - F_n(s) \leq \frac{C}{n} + \phi\left(\frac{\ell}{n}\right) - \phi(s)$$

and

$$-C\left(t - \frac{k}{n}\right) \leq F_n(t) - F_n\left(\frac{k}{n}\right) \leq C\left(t - \frac{k}{n}\right) + \phi(t) - \phi\left(\frac{k}{n}\right),$$

where $\phi(t) = \mathbb{P}(\eta \leq t)$ is the distribution function of η.

Proof. Recall (A.5) and (A.9). From an easy adaptation of the argument of the proof of Lemma A.4 and from Lemma A.7, we have

$$\begin{aligned}
\mathbb{E}\left(e^{[\Lambda_n(\ell)-\Lambda_n(\ell-1,s)]^-}-1\right) &\leq F_n\left(\frac{\ell}{n}\right)-F_n(s)\\
&\leq \mathbb{P}\left(s<\eta\leq\frac{\ell}{n}\right)+\mathbb{E}\left(e^{[\Lambda_n(\ell)-\Lambda_n(\ell-1,s)]^+}-1\right),\\
-\mathbb{E}\left([\Lambda_n(\ell)-\Lambda_n(\ell-1,s)]^-\right) &\leq F_n\left(\frac{\ell}{n}\right)-F_n(s)\\
&\leq \phi\left(\frac{\ell}{n}\right)-\phi\left(s\right)+C\mathbb{E}\left([\Lambda_n(\ell)-\Lambda_n(\ell-1,s)]^+\right),\\
-\frac{C}{n}-C\left(\frac{\ell}{n}-s\right) &\leq F_n\left(\frac{\ell}{n}\right)-F_n(s)\leq\frac{C}{n}+\phi\left(\frac{\ell}{n}\right)-\phi\left(s\right).
\end{aligned}$$

In the same way, we get the other assertion. □

We can now turn to proving A.3.

Proof of Proposition A.3. By combining Lemmas A.6 and A.8 and the fact that

$$F_n(t)-F_n(s)=F_n(t)-F_n\left(\frac{k}{n}\right)+F_n\left(\frac{k}{n}\right)-F_n\left(\frac{\ell}{n}\right)+F_n\left(\frac{\ell}{n}\right)-F_n(s),$$

we deduce that

$$-\frac{C}{n}-C(t-s)\leq F_n(t)-F_n(s)\leq C\left(t-\frac{\ell}{n}\right)+\phi(t)-\phi(s)+\frac{C}{n}.$$

Therefore,

$$-\frac{C}{n}-C(t-s)\leq F_n(t)-F_n(s)\leq C(t-s)+\phi(t)-\phi(s)+\frac{C}{n}.$$

The desired result follows. □

Recall that the goal of this appendix is to prove the convergence of the sequence $(F_n)_{n\geq 1}$ towards F, the unique solution of equation (3.2). For $T>0$, we define $w'_T(x,\cdot)$, the modulus of continuity of $x\in D([0,+\infty))$ on the interval $[0,T]$, by

$$w'_T(x,\delta)=\inf\max_{0\leq i<m}\sup_{t_i\leq s<t\leq t_{i+1}}|x(t)-x(s)|,$$

where the infimum is taken over the set of all increasing sequences $0=t_0<t_1<\cdots<t_m=T$ with the property that $\inf_{0\leq i<m}|t_{i+1}-t_i|\geq\delta$. Let $\{x_n:n\geq 1\}$ be a sequence function in $D([0,+\infty))$. The following result is a version of Theorem 12.3 from [2].

Proposition A.9. *Let $T > 0$. A necessary and sufficient condition for the sequence $\{x_n : n \geq 1\}$ to be relatively compact in $D([0, +\infty))$ is that the following hold:*

(i) $\sup_{n\geq 1} \sup_{0\leq t\leq T} |x_n(t)| < +\infty$,

(ii) $\lim_{\delta\to 0} \limsup_{n\to+\infty} w'_T(x_n, \delta) = 0$.

We now show that the sequence $(F_n)_{n\geq 1}$ satisfies the assertions of the above proposition.

Proposition A.10. *The sequence $(F_n)_{n\geq 1}$ is relatively compact in $D([0, +\infty))$.*

Proof. Condition (i) follows from Lemma A.2. Hence, it suffices to verify (ii). To this end, let us define $\psi(t) = \phi(t)+Ct$ for all $t > 0$, where again ϕ is the distribution function on η. It follows from Proposition A.3 that

$$|F_n(t) - F_n(s)| \leq \psi(t) - \psi(s) + \frac{C}{n} \quad \text{for all } t > s > 0. \tag{A.10}$$

It is easy to deduce from the definition of $w'_T(\cdot, \cdot)$ and (A.10) that

$$w'_T(F_n, \delta) \leq w'_T(\psi, \delta) + \frac{C}{n}.$$

Note that, since $\psi \in D([0, +\infty))$, $w'_T(\psi, \delta) \to 0$ as $\delta \to 0$ (see [2, Sect. 12, p. 123]). Thus, the desired result follows. □

We are now ready to state the main result of this appendix.

Proposition A.11. *As $n \to +\infty$, $\{F_n(t) : t > 0\} \to \{F(t) : t > 0\}$ in $D([0, +\infty))$, where F is the unique solution of* (3.2).

Proof. From Proposition A.10, we deduce that at least along a subsequence (but we use the same notation for the subsequence as for the sequence), F_n converges towards a limit denoted by J, where J is continuous on the right and admits a limit on the left. In order to show that $F = J$, it suffices to prove that J is a solution of equation (3.2) and then use Proposition 3.3. Indeed, let us rewrite (A.2) in the form

$$F_n(t) = \mathbb{E}\left[\mathbb{1}_{\eta\leq t} \exp\left\{ \int_0^{\frac{\lfloor nt\rfloor -1}{n}} \left(F_n\left(\frac{\lfloor nt\rfloor - \lceil nu\rceil}{n} \right) - 1 \right) \lambda(u)\, du \right\} - \int_{\frac{\lfloor nt\rfloor -1}{n}}^{t} \lambda(u)\, du \right].$$

Thus, it only remains to show that

$$F_n(t) \to J(t) = \mathbb{E}\left[\mathbb{1}_{\eta\leq t} \exp\left\{ \int_0^t \Big(J(t-u) - 1 \Big) \lambda(u)\, du \right\}\right] \quad \text{as } n \to +\infty$$

to obtain the desired result. To this end, we note that

$$\begin{aligned}\int_0^t \left| F_n\left(\frac{\lfloor nt\rfloor - \lceil nu\rceil}{n}\right) - J(t-u)\right| \lambda(u)\,du \\ &\le \int_0^t \left| F_n\left(\frac{\lfloor nt\rfloor - \lceil nu\rceil}{n}\right) - F_n(t-u)\right| \lambda(u)\,du \\ &\quad + \int_0^t |F_n(t-u) - J(t-u)|\lambda(u)\,du \\ &\le \lambda^* \int_0^t \psi(t-u) - \psi\left(t-u-\frac{2}{n}\right)\,du \\ &\quad + \frac{C}{n}\lambda^* t + \lambda^* \int_0^t |F_n(t-u) - J(t-u)|\,du,\end{aligned}$$

where we have used (A.10) in the last inequality. Since ψ is left-continuous and locally bounded, the first term tends to 0 as $n \to \infty$, thanks to Lebesgue's dominated convergence theorem. The second term tends clearly to 0. From the convergence in D for the Skorohod topology, $F_n(t-u) \to J(t-u)\,du$ a.e. Moreover, Lemma A.2 allows us to use Lebesgue's dominated convergence theorem again, and the result follows. □

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