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DIFFERENTIAL GRADED BRAUER GROUPS

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Abstract. We consider central simple K-algebras which happen to be differential graded K-algebras. Two such algebras A and B are considered equivalent if there are bounded complexes of finite-dimensional K-vector spaces C_A and C_B such that the differential graded algebras $A \otimes_K \operatorname{End}^\bullet_K(C_A)$ and $B \otimes_K \operatorname{End}^\bullet_K(C_B)$ are isomorphic. Equivalence classes form an abelian group, which we call the dg Brauer group. We prove that this group is isomorphic to the ordinary Brauer group of the field K.

Introduction

Brauer groups proved to be an important invariant of fields K. They are defined as equivalence classes of central simple K algebras, where two such algebras are called equivalent if a matrix algebra of the one is isomorphic to the matrix algebra (of possibly different size) of the other. These then form an abelian group under tensor product over K. Most interestingly, for algebraic number fields, is the link to the Brauer groups of the completions at the primes of the field. The Brauer group then embeds into the product of the Brauer groups over all completions, with cokernel being isomorphic to $\mathbb{Q}/\mathbb{Z}$.

The original definition was generalised further to graded central simple K algebras (originally by C. T. C. Wall for a grading over the cyclic group of order 2, motivated by studies on quadratic forms), to Hopf algebras, and culminating in the maybe the most far reaching generalisation to braided monoidal categories, which was given by van Oystaen and Zhang [10]. Most recently, [4] developed a theory of Brauer groups for (non graded!) differential central simple algebras in the context of differential Galois theory. However, this theory is very different since skew fields never contain nilpotent elements, and hence only trivial gradings on finite-dimensional skew fields are possible. Therefore, skew fields only bear a trivial differential graded structure.

In [12] we studied differential graded orders in differential graded algebras which are semisimple as algebras. In particular we studied the local-global behaviour and defined a theory of idèles of theses structures. It seems to be natural then to

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consider Brauer groups for differential graded algebras which are central simple as algebras. This is what we propose to do in the present note.

There are two possible versions. First we may consider differential graded K-algebras which are central simple as algebras, which leads to what we call $\mathrm{dgBr}(K)$, the differential graded Brauer group.

Another option is the construction given by algebras which are finite-dimensional differential graded K-algebras and whose category of differential graded modules is semisimple. Aldrich and Garcia-Rozas [1] proved a structure theorem for these. They are formed by differential graded algebras (A, d), which are acyclic as complexes, and such that $\ker(d)$ is central $\mathbb{Z}$-graded simple. Moreover, in this case for any $z \in d^{-1}(1)$ we have $A = \ker(d) \oplus z\ker(d)$. This concept does not give a Brauer group since the tensor product of two such algebras will in general not be simple in the sense of [1], and actually not even semisimple. We shall provide a counterexample.

Our main result is the proof that the forgetful functor induces a group isomorphism $\mathrm{dgBr}(K) \simeq \mathrm{Br}(K)$.

The paper is organized as follows. In Section 1 we recall the definitions and notations which we use for differential graded algebras. Section 2 then recalls the definition of the ordinary Brauer group of central simple algebras, the definition of Brauer groups in the graded sense, and gives our main definition, namely the Brauer group of differential graded algebras and shows first properties. We also provide an example why the theory of semisimplicity in the category of differential graded algebras given by [1] is not well suited for our purposes. Section 3 then states and proves our main result.

1. Foundations of dg-algebras and dg-modules

We recall some definitions concerning differential graded algebras and their differential graded modules, also to fix the notations.

Let R be a commutative ring. Recall from Cartan [2], Keller [6], and [8] that:

(1) A differential graded R-algebra (or dg-algebra for short) is a $\mathbb{Z}$-graded R-algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ together with a graded R-linear endomorphism d satisfying $d(A_n) \subseteq A_{n+1}$ and $d \circ d = 0$, and such that

$$d(ab) = d(a)b + (-1)^{|a|}ad(b)$$

for all homogeneous elements $a, b \in A$. Here $|a|$ denotes the degree of a. A homomorphism of differential graded algebras $f : (A, d_A) \to (B, d_B)$ is a degree 0 homogeneous R-algebra map f such that $f \circ d_A = d_B \circ f$.

(2) For differential graded K-algebras (A, d_A) and (B, d_B), also $A \otimes_K B$ is a differential graded algebra. The differential is defined by

$$d_{A\otimes_K B}(a \otimes b) = d_A(a) \otimes b + (-1)^{|a|}a \otimes d_B(b)$$

for homogeneous elements a and b.

(3) If (A, d) is a differential graded algebra, then (A^{op}, d^{op}) is a differential graded algebra (cf. e.g. [8, Tag 09JG]) with $x \cdot_{op} y := (-1)^{|x|\cdot|y|}yx$ for

any homogeneous elements $x, y \in A$, and $d^{op}(x) = d(x)$. We hence write $d^{op} = d$.

(4) A differential graded right A-module (or dg-module for short) is then a $\mathbb{Z}$-graded R-module M with graded R-linear endomorphism d_M of square 0 and degree 1, such that

$$d_M(ma) = d_M(m)a + (-1)^{|m|} m d(a)$$

for all homogeneous elements $a \in A$ and $m \in M$. A differential graded (A, d)-left module is a differential graded (A^{op}, d)-right module.

(5) Let (A, d_A) be a differential graded R-algebra and let (M, δ_M) and (N, δ_N) be differential graded (A, d_A)-modules. Then a homomorphism of differential graded modules is an R-linear homogeneous map $f : M \to N$ of degree 0 with $f \circ \delta_M = \delta_N \circ f$, with $f(am) = af(m)$ for all $a \in A$ and $m \in M$.

(6) Let (M, d_M) and (N, d_N) be differential graded (A, d)-modules. The homomorphism complex $\mathrm{Hom}_A^\bullet(M, N)$ is the $\mathbb{Z}$-graded R-module given by

$$(\mathrm{Hom}_A^\bullet(M, N))_n := \{f : M \to N \mid f \in \mathrm{Hom}_{A\text{-graded}}(M, N) \text{ and } f(M_k) \subseteq N_{k+n}\}.$$

The elements f of $\mathrm{Hom}_A^\bullet(M, N)$ are not asked to be compatible with the differentials in any way. Let $d_{\mathrm{Hom}} : \mathrm{Hom}_A^\bullet(M, N) \to \mathrm{Hom}_A^\bullet(M, N)$ given by

$$d_{\mathrm{Hom}}(f) := d_N \circ f - (-1)^{|f|} f \circ d_M.$$

Then $d_{\mathrm{Hom}}^2(f) = 0$, as is easily verified (cf. e.g. [12]). Hence the pair $(\mathrm{Hom}_A^\bullet(M, N), d_{\mathrm{Hom}})$ is a complex of R-modules. Moreover, in case $M = N$ we get that $(\mathrm{Hom}_A^\bullet(M, M), d_{\mathrm{Hom}})$ is a differential graded algebra.

2. Definition of differential graded Brauer groups

We recall the Brauer group of a field.

Definition 2.1. Let K be a field. Two finite-dimensional central simple K-algebras A and B are equivalent if there are positive integers $m, n \in \mathbb{N}$ such that $A \otimes_K \mathrm{Mat}_n(K) \simeq B \otimes_K \mathrm{Mat}_m(K)$. The *Brauer group* $\mathrm{Br}(K)$ is the group with elements being the equivalence classes of finite-dimensional central simple K-algebras and group law induced by $- \otimes_K -$.

Recall that this is indeed a group. It is a set since we consider equivalence classes. The law is clearly well defined and associative. The neutral element is the equivalence class of K. Further, for any finite-dimensional central simple K-algebra A we have a classical result (cf. e.g. [5, Theorem 4.1.3]), which shows that $A \otimes_K A^{op} \simeq \mathrm{Mat}_{\dim_K(A)}(K)$. Hence the inverse of the equivalence class of A is the equivalence class of A^{op}.

For the so-called G-graded Brauer group $\mathrm{GBr}(K)$, for the (possibly infinite) cyclic group G, an analogous construction is used. Note however that the graded Brauer group is slightly different from the one in Definition 2.7.

A G-graded K-algebra A is called graded central simple if A does not have non trivial G-graded twosided ideals and $K = Z(A)$. If A and B are two graded central simple K-algebras, then $A \otimes_K B$ is again a graded central simple K-algebra. Note that here the tensor product is the graded tensor product defined as follows. As a K-module, this is the usual tensor product. However, the multiplication law is given by $(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$ for homogeneous elements a, b, c, d.

Again, it can be shown that $A \otimes_K A^{op} \simeq \mathrm{End}_K(A)$ as a graded algebra. As in the ungraded case one defines an abelian group, the G-graded Brauer group (cf. e.g. Turbow [9]).

Definition 2.2. Let K be a field and let G be a cyclic group. Two central G-graded simple K-algebras A and B are equivalent if there are G-graded K-modules V and W such that $A \otimes_K \mathrm{End}_K(V) \simeq B \otimes_K \mathrm{End}_K(W)$. Then the *$G$-graded Brauer group* $\mathrm{GBr}(K)$ is the group of equivalence classes of central G-graded simple K-algebras.

For the G-graded Brauer group, one considers algebras which are simple as G-graded algebras. Algebras which are simple as G-graded algebras need not be simple as algebras. However, simple algebras, which happen to be G-graded, are of course graded simple. Obviously simple algebras are graded simple with the trivial grading. The graded Brauer group is well studied, and in case G is or order 2, this group is called the Brauer–Wall group, after C. T. C. Wall's work [11].

2.1. **General properties of differential graded algebras.** Our intention is to define a Brauer group for differential graded algebras.

We have (at least) two concepts for what we should call a simple differential graded algebra. The somehow naive version, but underpinned by the success of the concept of differential graded orders in [12] consists in considering central simple K-algebras A, which are in addition differential graded. Our objects then would be such algebras (A, d).

A second more categorical concept would be to consider indecomposable differential graded algebras whose category of differential graded modules is semisimple.

Aldrich and Garcia-Rochas showed in [1] that a differential graded algebra (A, d) has a semisimple category of differential graded modules if and only if (A, d) is bounded and acyclic, and moreover $\ker(d)$ is semisimple as an ordinary graded algebra. Further, in this case, for $z \in A$ with $d(z) = 1$, we have $A = \ker(d) \oplus z \cdot \ker(d)$. Observe that for any homogeneous element $n \in \ker(d)$ we have

$$d(zn) = d(z)n + (-1)^{|z|}z \cdot d(n) = 1 \cdot n + 0 = n.$$

Note further that we may use the same element z also for the opposite algebra.

Example 2.3. For a field K consider $A = K[X]/X^2$ where K is in degree 0 and X is an element of degree -1. Then there is a differential d on $K[X]/X^2$ given by $d(1) = 0$ and $d(X) = 1$. Note that (A, d) is acyclic and $\ker(d) = K$ is simple. Hence (A, d) is semisimple in the sense of [1], but A is not semisimple as an algebra.

On the other hand, the field K in degree 0, and differential 0 is a simple algebra, which is differential graded by the trivial dg-structure. However, it is not semisimple in the sense of [1].

We hence first need an analogue for what provides the inverse of a central simple algebra in the classical case.

Proposition 2.4. *Let K be a field. Let (A,d) be a differential graded K-algebra which is central simple as an algebra. Then $(A,d)\otimes_K(A^{op},d_A^{op})\simeq(\mathrm{End}_K^\bullet(A),d_{\mathrm{Hom}})$, and hence $(A,d)\otimes_K(A^{op},d_A^{op})\simeq(\mathrm{Mat}_{\dim_K(A)}(K),d_M)$ for some grading and differential d_M on the matrix algebra.*

Proof. As (A,d) is a (the regular) (A,d)-module, by the preliminary remarks, (A,d) is a right (A^{op},d)-module as well. Hence, considering (A,d) as a bounded complex (C,d) of K-modules, we consider left multiplication

$$\lambda:(A,d)\longrightarrow(\mathrm{End}_K^\bullet(C),d_{\mathrm{Hom}})$$

given by $\lambda_a(x)=ax$ for any $a\in(A,d)$ and $x\in(C,d)$ as well as right multiplication given by

$$\rho:(A^{op},d)\longrightarrow(\mathrm{End}_K^\bullet(C),d_{\mathrm{Hom}})$$

given by $\rho_a(x)=a\cdot_{op}x$ for any $a\in(A^{op},d)$ and $x\in(C,d)$. These are clearly algebra homomorphisms. We need to see that these are dg-homomorphisms between dg-algebras.

We first need to show $\lambda_{d(a)}=d_{\mathrm{Hom}}(\lambda_a)$ for all homogeneous $a\in A$. This then translates into

$$\begin{aligned}d_{\mathrm{Hom}}(\lambda_a)(b)&=(d\circ\lambda_a-(-1)^{|a|}\lambda_a\circ d)(b)\\&=d(ab)-(-1)^{|a|}a\cdot d(b)\\&=d(a)\cdot b\\&=\lambda_{d(a)}(b)\end{aligned}$$

for all $b\in A$.

Then, we need to show $\rho_{d(a)}=d_{\mathrm{Hom}}(\rho_a)$ for all homogeneous $a\in A^{op}$. This then becomes

$$\begin{aligned}d_{\mathrm{Hom}}(\rho_a)(b)&=(d\circ\rho_a-(-1)^{|a|}\rho_a\circ d)(b)\\&=d(\rho_a(b))-(-1)^{|a|}\rho_a(d(b))\\&=d(a\cdot_{op}b)-(-1)^{|a|}a\cdot_{op}d(b)\\&=(-1)^{|b||a|}d(ba)-(-1)^{|a|}\cdot(-1)^{|a|\cdot(|b|+1)}d(b)a\\&=(-1)^{|b||a|}(d(ba)-(-1)^{|a|+|a|}d(b)a)\\&=(-1)^{|b||a|}(d(ba)-d(b)a)\\&=(-1)^{|b||a|}\cdot(-1)^{|b|}bd(a)\\&=(-1)^{|b|\cdot(|a|+1)}bd(a)\\&=d(a)\cdot_{op}b\\&=\rho_{d(a)}(b).\end{aligned}$$

Clearly ρ is injective, as well as λ. Hence

$$A\simeq\lambda(A)\subseteq(\mathrm{End}_K^\bullet(C),d_{\mathrm{Hom}})$$

and

$$A^{op} \simeq \rho(A^{op}) \subseteq (\mathrm{End}_K^\bullet(C), d_{\mathrm{Hom}}).$$

Now if, as an algebra, A is simple, also $\lambda(A)$ is simple as an algebra. Likewise as an algebra, A^{op} is simple, and hence also $\rho(A^{op})$. A classical result [5, Theorem 4.1.1] then shows that $\lambda(A) \otimes_K \rho(A^{op})$ is simple. But now the map

$$A \otimes_K A^{op} \simeq \lambda(A) \otimes_K \rho(A^{op}) \longrightarrow \lambda(A) \cdot \rho(A^{op}) \subseteq (\mathrm{End}_K^\bullet(C), d_{\mathrm{Hom}})$$

is necessarily injective, by the simplicity of $A \otimes_K A^{op}$.

Now, for $n = \dim_K(A)$ we have

$$n^2 = \dim_K(A \otimes_K A^{op}) = \dim_K(\lambda(A) \cdot \rho(A^{op})) \leq \dim_K(\mathrm{End}_K^\bullet(C, d)) = n^2$$

again, and hence

$$A \otimes_K A^{op} \simeq \lambda(A) \cdot \rho(A^{op}) = \mathrm{End}_K^\bullet(C, d)$$

as differential graded algebras. This shows the proposition. □

This motivates the following definition.

Definition 2.5. Two differential graded K-algebras A and B, are called *equivalent* if there are bounded complexes C_1 and C_2 of finite-dimensional K-modules such that

$$A \otimes_K \mathrm{End}_K^\bullet(C_1) \simeq B \otimes_K \mathrm{End}_K^\bullet(C_2).$$

We prove now directly the following lemma.

Lemma 2.6. *Let K be a field and let (A, d_A) and (B, d_B) be differential graded algebras. Then $(A \otimes_K B, d_{A \otimes_K B}) \simeq (B \otimes_K A, d_{B \otimes_K A})$ as differential graded algebras.*

Proof. We need to show commutativity of the graded tensor product. Let A and B be differential graded K-algebras. Then $A \otimes_K B$ has multiplication

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|}(aa' \otimes bb').$$

The algebra $B \otimes_K A$ has multiplication

$$(b \otimes a) \cdot (b' \otimes a') = (-1)^{|a||b'|}(bb' \otimes aa').$$

Then,

$$\begin{aligned} A \otimes_K B &\xrightarrow{\alpha} B \otimes_K A \\ a \otimes b &\longmapsto (-1)^{|a||b|}(b \otimes a) \end{aligned}$$

for any homogeneous elements $a, a' \in A$, $b, b' \in B$ is an algebra isomorphism. Indeed, for any homogeneous elements $a, a' \in A$ and $b, b' \in B$,

$$\begin{aligned}
\alpha((a \otimes b) \cdot (a' \otimes b')) &= \alpha((-1)^{|b||a'|}(aa' \otimes bb')) \\
&= (-1)^{|aa'||bb'|} \cdot (-1)^{|b||a'|}(bb' \otimes aa') \\
&= (-1)^{(|a|+|a'|)(|b|+|b'|)} \cdot (-1)^{|b||a'|}(bb' \otimes aa') \\
&= (-1)^{|a||b|+|a||b'|+|a'||b'|}(bb' \otimes aa') \\
&= (-1)^{|a||b|+|a'||b'|}(b \otimes a) \cdot (b' \otimes a') \\
&= (-1)^{|a||b|}(b \otimes a) \cdot (-1)^{|a'||b'|}(b' \otimes a') \\
&= \alpha(a \otimes b) \cdot \alpha(a' \otimes b')
\end{aligned}$$

We need to show compatibility with the differential.

$$\begin{aligned}
\alpha(d(a \otimes b)) &= \alpha(da \otimes b + (-1)^{|a|} a \otimes db) \\
&= (-1)^{|b|(|a|+1)}(b \otimes da) + (-1)^{|a|} \cdot (-1)^{|a|(|b|+1)}(db \otimes a) \\
&= (-1)^{|a||b|}(db \otimes a + (-1)^{|b|} b \otimes da) \\
&= d(\alpha(a \otimes b))
\end{aligned}$$

This shows the lemma. ☐

2.2. **The differential graded Brauer group.** Recall the notion of equivalent simple differential graded algebras from Definition 2.5.

Definition 2.7. The *dg-Brauer group* $\mathrm{dgBr}(K)$ *of a field* K is given by the set of equivalence classes (in the sense of Definition 2.5) of algebras A, which are central simple as K-algebras and which are in addition differential graded algebras.

Actually, the notion of equivalence from Definition 2.5 used for Definition 2.7 seems to be a little too strong. We only need to consider complexes C which are the K-module structure of central simple dg K-algebras.

One might be tempted to define another dg Brauer group by simple dg-algebras in the sense of Aldrich and Garcia-Rochas [1]. This would then possibly be linked to the graded Brauer group, which is very well studied.

Example 2.8. Recall Example 2.3. Let K be a field of characteristic 2 and let $A = K[X]/X^2$ where X is in degree -1, where $d(X) = 1$, and $d(1) = 0$. Then

$$A \otimes_K A = K[X]/X^2 \otimes K[Y]/Y^2 = K[X, Y]/(X^2, Y^2)$$

is a differential graded algebra concentrated in degrees 0, -1 and -2. We get that the differential D on the tensor product is $D(XY) = X - Y$ and $D(X) = 1 = D(Y)$. Further, $D(1) = 0$. This shows that we have $\ker(D) = K + K(X - Y)$ and hence $\ker(D) = K[Z]/Z^2$ where $Z = X - Y$ is of degree -1. This algebra is not semisimple, whereas $\ker(d) = K$ is semisimple. This shows that $(A, d) \otimes_K (A, d)$ is not simple as a differential graded algebra in the sense of [1], even though (A, d) is simple in the sense of [1]. It is therefore impossible to define a Brauer group for this class of algebras in the same way as it is done classically. Our observation here confirms our observation from [12] that the concept of semisimple differential graded algebras in the sense of [1] is much less well behaved as our concept.

Proposition 2.9. $\mathrm{dgBr}(K)$ *is an abelian group with group law the graded tensor product and the inverse element of a class being the equivalence class of the opposite algebra.*

Proof. The tensor product is easily seen to be well defined. Denote by $[A]$ the equivalence class of a central simple differential graded algebra A. The equivalence class $[K]$ of the 1-dimensional algebra K concentrated in degree 0 and differential 0 is the neutral element of $\mathrm{dgBr}(K)$. By Proposition 2.4 for each element $[A]$ of $\mathrm{dgBr}(K)$ we have that

$$[A] \cdot [A^{op}] = [A \otimes_K A^{op}] = [K].$$

Hence all elements of $\mathrm{dgBr}(K)$ have an inverse. The associativity of the group law is a general property of the tensor product of differential graded algebras. By Lemma 2.6 the group law is commutative.

We hence have proved the proposition. □

3. The main result

In [12] we proved a structure theorem for split simple dg-algebras. In the proof of the theorem we needed a technical hypothesis, namely that there is a primitive idempotent e of A such that $A \cdot e \not\subseteq A \cdot d(e)$. In a more general setting this may be false, as illustrated by the following example.

Example 3.1. Recall Example 2.3. Let K be a field, and consider the graded algebra $A = K[X]/X^2$ where K is in degree 0 and X is an element of degree -1. Then there is a differential d on $K[X]/X^2$ given by $d(1) = 0$ and $d(X) = 1$ such that (A, d) is a differential graded algebra. Note that here we have $AX \subseteq Ad(X)$. Of course, A is not semisimple and X is not idempotent.

However, for full matrix algebras over fields, the hypothesis $A \cdot e \not\subseteq A \cdot d(e)$ in the above mentioned structure theorem from [12] is superfluous, as we shall prove now.

Recall the following result by Dascalescu, Ion, Nastasescu, and Rios-Montes from [3]. Consider the full matrix algebra $\mathrm{End}_K(K^n)$ over a field K and denote by $e_{i,j}$ the matrix that has coefficient 0 everywhere except at position (i, j), where it has coefficient 1. The authors of [3] call a group grading on a full matrix algebra $\mathrm{Mat}_n(K)$ *good* if the matrices $e_{i,j}$ are homogeneous elements of the grading.

Theorem 3.2. *The following statements hold.*

- ([3, Theorem 1.4]). *Let R be the algebra $\mathrm{Mat}_n(K)$ endowed with a G-grading such that there is a G-graded R-module which is simple as an R-module. Then there exists an isomorphism of graded algebras $R \simeq S$ where S is $\mathrm{Mat}_n(K)$ endowed with a good grading.*

- ([3, Corollary 1.5]). *If G is torsion free, then any grading on* $\mathrm{Mat}_n(K)$ *is isomorphic to a good grading.*
- ([3, Proposition 2.1]). *There is a bijective correspondence between the set of all good G-gradings on* $\mathrm{Mat}_n(K)$ *and the set of maps $f : \{1, 2, \dots, n-1\} \longrightarrow G$ such that to a good G-grading we associate the map defined by $f(i) = \deg(e_{i,i+1})$.*

We should mention that [3] also provides examples of non good gradings on $\mathrm{Mat}_n(K)$.

The following proposition was proved in [12] under an additional technical assumption. This assumption is superfluous, as we shall prove now. For the convenience of the reader we also recall all the details of the proof from [12] in order to have a complete presentation.

Proposition 3.3. *Let K be a field and let (A, d) be a finite-dimensional differential graded K-algebra. Suppose that A is a split simple K-algebra. Then there is a bounded complex L of K-modules such that $A \simeq \mathrm{Hom}^\bullet_K(L, L)^{op}$ as differential graded algebras. Conversely, $A = \mathrm{Hom}^\bullet_K(L, L)$ is differential graded, finite-dimensional simple as algebra.*

Proof. If L is a bounded complex of K-vector spaces, then $\mathrm{Hom}^\bullet_K(L, L)$ is a full matrix ring over K, as ungraded algebra, and hence simple as algebra. Further, as recalled from Section 1 item (6), the algebra $\mathrm{Hom}^\bullet_K(L, L)$ is a differential graded algebra.

Conversely, let K be a field and let (A, d) be a finite-dimensional differential graded algebra. Suppose that A is a split simple K-algebra. By Wedderburn's theorem, A is a full matrix algebra over K. Let e be a primitive idempotent of A.

By Theorem 3.2 we may assume that A is $\mathbb{Z}$-graded by a good grading, and $e_{i,i}$ are all of degree 0. As d is of degree 1, we may choose $e = e_{i,i}$ (depending on whether the degree 1 element is upper or lower diagonal) such that $Ae \not\subseteq Ad(e)$.

Since $Ae \not\subseteq Ad(e)$,

$$M := A \cdot e + A \cdot d(e) \quad \text{and} \quad N := A \cdot d(e)$$

are differential graded (A, d)-modules. Further $N < M$ and

$$L := M/N \neq 0$$

is a differential graded (A, d)-module.

As an A-module, we see that $L \simeq Ae$ is a progenerator. Hence L is a natural differential graded (A, d)-$(\mathrm{End}^\bullet_K(L), d_{\mathrm{Hom}})$ bimodule. Now, for any homogeneous $a \in A$, left multiplication by a gives a homogeneous element $\varphi(a) \in \mathrm{End}^\bullet_K(L)$. Further, φ is additive, sends $1 \in A$ to the identity on L, and induces a ring homomorphism

$$\varphi : A \longrightarrow \mathrm{End}^\bullet_K(L)^{op}.$$

Since L is a progenerator, φ is injective. Since $\dim_K(A) = \dim_K(\mathrm{End}^\bullet_K(L))$, we get that φ is an isomorphism of algebras. Now, for any homogeneous $a, b \in A$, we have

$$d(a)b = d(ab) - (-1)^{|a|} a d(b),$$

so we get

$$\varphi(d(a)) = d \circ \varphi(a) - (-1)^{|a|}\varphi(a) \circ d = d_{\mathrm{Hom}}(\varphi(a))$$

and therefore φ is an isomorphism of differential graded algebras. □

Remark 3.4. After completing and submitting the manuscript I discovered that D. Orlov had previously defined our notion of a simple differential graded algebra in [7], where he referred to it as *abstractly simple*. Moreover, he proved the statement of Proposition 3.3 by entirely different means, employing scheme-theoretic arguments. However, his proof requires the assumption that the primitive central idempotents are in degree 0. In contrast, our approach gives that this can be assumed to be automatically satisfied using [3, Corollary 1.5].

Theorem 3.5. *Let K be a field. Then the forgetful functor induces an isomorphism*

$$\mathrm{Br}(K) \simeq \mathrm{dgBr}(K).$$

Proof. We obviously have a group homomorphism

$$\mathrm{Br}(K) \xrightarrow{\iota} \mathrm{dgBr}(K)$$

since any central simple algebra is also a central simple dg-algebra with trivial grading and 0 differential. Further, the map induced by just forgetting the grading and the differential induces a group homomorphism

$$\mathrm{dgBr}(K) \xrightarrow{\phi} \mathrm{Br}(K).$$

Of course,

$$\phi \circ \iota = \mathrm{id}_{\mathrm{Br}(K)}.$$

Hence, ϕ is surjective. Consider $\ker(\phi)$. By definition, $\ker(\phi)$ is formed by equivalence classes of differential graded central simple algebras A such that $A \otimes_K \mathrm{Mat}_n(K) \simeq \mathrm{Mat}_m(K)$ as ungraded algebras, for some $n, m \in \mathbb{N}$. But this shows that $m = n \cdot \dim_K(A)$ and

$$\mathrm{Mat}_n(A) \simeq \mathrm{End}_K(K^{n \cdot \dim_K(A)}).$$

By Proposition 3.3 there is a complex C in $C^b(K\text{-mod})$ such that we have an isomorphism of differential graded algebras

$$\mathrm{Mat}_n(A) \simeq (\mathrm{End}_K^\bullet(C), d_{\mathrm{Hom}}).$$

Hence $[A] = [K]$ in $\mathrm{dgBr}(K)$. This shows that Φ is an isomorphism. Therefore,

$$\mathrm{dgBr}(K) \simeq \mathrm{Br}(K)$$

as claimed. □

Remark 3.6. We emphasize that Theorem 3.5 shows that if K is a field, then any central simple differential graded K-algebra A is equivalent to one of the form $\mathrm{End}_K^\bullet(C) \otimes_K D$ for some complex C in $C^b(K\text{-mod})$, and some finite-dimensional (ungraded) skew field D with centre K.

Remark 3.7. Since $\mathrm{Br}(K)$ is abelian, Theorem 3.5 shows that also $\mathrm{dgBr}(K)$ is abelian, without using Lemma 2.6. However, Lemma 2.6 is completely elementary, whereas Theorem 3.5 is not really. Further, we show in Lemma 2.6 commutativity of the tensor product in general, and not only up to equivalence in the dg-Brauer group as it follows from Theorem 3.5.

Remark 3.8. Let (A, d) and (B, d) be simple differential graded algebras in the sense of [1]. Since the differential of a tensor product of dg-algebras (A, d_A) and (B, d_B) is given as $(A \otimes B, d_A \otimes \mathrm{id}_B \pm \mathrm{id}_A \otimes d_B)$, we have the $\ker(d_A) \otimes \ker(d_B)$ is indeed a subspace of $\ker(d_A \otimes \mathrm{id}_B \pm \mathrm{id}_A \otimes d_B)$, but the kernel contains more in general coming from elements of the form $x_A \otimes z_B x_B \pm z_A x_A \otimes x_B$ for elements $x_A \in \ker(d_A)$ and $x_B \in \ker(d_B)$. Hence, taking the kernel of the differential will not satisfy that $(A, d_A) \otimes_K (B, d_B)$ maps to the class of $\ker(d_A) \otimes_K \ker(d_B)$ in $\mathrm{GBr}(K)$. Example 2.8 is formed in this sense and provides an explicit example for this phenomenon.

Remark 3.9. By Theorem 3.5 we have $\mathrm{Br}(K) \xrightarrow{\iota} \mathrm{dgBr}(K)$ is an isomorphism where $\iota([A]) = [(A, 0)]$. We may be tempted to consider for a differential graded algebra (A, d) its homology $H(A, d)$. Since by Künneth's formula we get

$$H((A, d) \otimes_K (\mathrm{End}^\bullet_K(C), d_{\mathrm{Hom}})) \simeq H((A, d) \otimes_K H(\mathrm{End}^\bullet_K(C), d_{\mathrm{Hom}}))$$

the map 'taking homology' from $\mathrm{dgBr}(K)$ to equivalence classes of graded modules is not really well defined. Indeed, $H(\mathrm{End}^\bullet_K(C), d_{\mathrm{Hom}})$ is in general not isomorphic to $\mathrm{End}_K(H(C))$. One would need to consider a broader concept of equivalence for a modified graded Brauer group at least. Further, even then, the isomorphism $\mathrm{Br}(K) \xrightarrow{\iota} \mathrm{dgBr}(K)$ and the fact that $H(A, 0) \simeq A$ indicates that homology is not an interesting map on the differential graded Brauer group.

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