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LORENTZ C_{12}-MANIFOLDS

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Abstract. The object of the present paper is to study C_{12}-structures on a manifold with Lorentzian metric. We focus here on Lorentzian C_{12}-structures, emphasizing their relationship and analogies with respect to the Riemannian case. Several interesting results are obtained. Next, we study Ricci solitons in Lorentzian C_{12}-manifolds.

1. Introduction

Recently, C_{12}-manifolds have become a well-known and intensively studied subject of research in differential geometry. The recent works [1, 2, 3, 4, 5] provide a detailed overview of the results obtained in this framework.

What distinguishes a C_{12}-manifold from other almost contact metric structures is that it is neither contact nor normal. It has important characteristics similar to well-known manifolds such as the Sasaki, Kenmotsu, and cosymplectic manifolds. That is why we find it important to study on it the various concepts that were studied on other previously mentioned manifolds, and compare the results obtained.

Although Lorentz manifolds are the main topic in physics, there may be some obstructions to the existence of a Lorentz metric on a manifold. A condition under which it is possible to construct a Lorentz metric from a Riemannian one is the existence of a globally-defined nowhere-vanishing vector field. Such a condition is clearly satisfied by a C_{12}-manifold; hence using O'Neill's construction we realize a Lorentz metric on a C_{12}-manifold and we study its properties.

Therefore, it is a very natural and interesting idea to define both a C_{12}-manifold structure and a Lorentzian metric on an odd-dimensional manifold.

First of all, we will start by introducing the basic concepts that we need in this research.

2. Preliminaries

The notion of Ricci soliton was introduced by Hamilton in 1982 [8]. A Ricci soliton is a natural generalization of an Einstein metric. A pseudo-Riemann manifold (M, g) is called a *Ricci soliton* if it admits a smooth vector field V (potential

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vector field) on M such that

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) - 2\lambda g(X,Y) = 0, \tag{2.1}$$

where $\mathcal{L}_V g$ is the Lie derivative of g along V given by

$$(\mathcal{L}_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_V Y, X),$$

S is the Ricci tensor, λ is a constant and X, Y are arbitrary vector fields on M.

A Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold with V zero or Killing.

The generalized Ricci soliton equation in a Riemann manifold (M,g) is defined by (see [9])

$$\mathcal{L}_V g = -2c_1 V^\flat \otimes V^\flat + 2c_2 S + 2\lambda g, \tag{2.2}$$

where $V^\flat(X) = g(V, XY)$ and $c_1, c_2, \lambda \in \mathbb{R}$.

A further generalization of the Ricci soliton equation in the Riemann manifold (M,g), is given by the following equation (see [6]):

$$\mathcal{L}_{V_1} g = -2c_1 V_2^\flat \otimes V_2^\flat + 2c_2 S + 2\lambda g, \tag{2.3}$$

where V_1, V_2 are two vector fields on M.

Recently, in [3], the authors introduced the generalized η-Ricci soliton equation in a Riemann manifold (M,g) given by

$$\mathcal{L}_V g = -2c_1 V^\flat \otimes V^\flat + 2c_2 S + 2\lambda g + 2\mu\eta \otimes \eta, \tag{2.4}$$

where $c_1, c_2, \lambda, \mu \in \mathbb{R}$ and η is a 1-form on M.

Inspired by equations (2.3) and (2.4), we can guess the existence of a generalization that includes all previous cases, which we define by the following equation:

$$\mathcal{L}_{V_1} g = -2c_1 V_2^\flat \otimes V_2^\flat + 2c_2 S + 2\lambda g + 2\mu\eta \otimes \eta. \tag{2.5}$$

We refer to this generalization as "generalized η-Ricci bi-soliton" and the confirmation of the existence of this generalization will be in the last two theorems. An odd-dimensional Riemann manifold (M^{2n+1}, g) is said to be an *almost contact manifold* if there exist on M a $(1,1)$-tensor field φ, a vector field ξ (called the *structure vector field*) and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2 X = -X + \eta(X)\xi \end{cases} \tag{2.6}$$

for any vector fields X, Y on M.

In particular, in an almost contact metric manifold we also have

$$\varphi\xi = 0 \qquad \text{and} \qquad \eta \circ \varphi = 0.$$

Moreover, a Riemannian metric is said to be compatible with the almost contact structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

and we refer to an almost contact metric structure as (φ, ξ, η, g).

The fundamental 2-form ϕ is defined by

$$\phi(X,Y)=g(X,\varphi Y).$$

It is known that the almost contact structure (φ,ξ,η) is said to be normal if and only if

$$N^{(1)}(X,Y)=N_\varphi(X,Y)+2d\eta(X,Y)\xi=0$$

for any X, Y on M, where N_φ denotes the Nijenhuis torsion of φ, given by

$$N_\varphi(X,Y)=\varphi^2[X,Y]+[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y].$$

Let now $\tilde{g}$ denote a Lorentz metric on M; $\tilde{g}$ is said to be compatible with the almost contact structure (φ,ξ,η) if

$$\tilde{g}(\varphi X,\varphi Y)=\tilde{g}(X,Y)+\eta(X)\eta(Y). \tag{2.7}$$

A smooth manifold M, equipped with an almost contact structure (φ,ξ,η) and a compatible Lorentz metric $\tilde{g}$, will be called an *almost contact Lorentz manifold.*

Note that, by (2.6) and (2.7), $\eta(X)=-\tilde{g}(\xi,X)$. In particular, $\tilde{g}(\xi,\xi)=-1$, and so the characteristic vector field ξ is timelike with respect to the metric $\tilde{g}$. Moreover, (2.7) implies that $\tilde{g}(\varphi X,Y)=-\tilde{g}(X,\varphi Y)$.

In the classification of Chinea and Gonzalez [7] of almost contact metric manifolds there is a class called C_{12}-manifolds which can be integrable but never normal.

Definition 2.1 ([4])**.** Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost contact manifold. M is called an *almost C_{12}-manifold* if there exists a closed 1-form w on M that satisfies

$$\mathrm{d}\eta=\omega\wedge\eta,\qquad \mathrm{d}\phi=0.$$

In addition, if $N_\varphi=0$, we say that M is a *C_{12}-manifold* and is denoted by $(M^{2n+1},g,\xi,\psi,\eta,\omega,g)$, where $\omega=-\big(\nabla_\xi\xi\big)^\flat=-\nabla_\xi\eta$ and ψ is the vector field given by

$$\omega(X)=g(X,\psi)=-g(X,\nabla_\xi\xi)$$

for all vector fields X on M.

In a C_{12}-manifold, the following conditions are equivalent (see [2, 4, 5, 7]):

$$\begin{aligned}(\nabla_X\phi)(Y,Z)&=\eta(X)\eta(Z)(\nabla_\xi\eta)\varphi Y-\eta(X)\eta(Y)(\nabla_\xi\eta)\varphi Z,\\ (\nabla_X\varphi)Y&=\eta(X)\big(\omega(\varphi Y)\xi+\eta(Y)\varphi\psi\big),\\ (\nabla_{\varphi X}\varphi)Y&=0.\end{aligned} \tag{2.8}$$

Putting $Y=\xi$ in (2.8), one easily obtains

$$\nabla_X\xi=-\eta(X)\psi. \tag{2.9}$$

Proposition 2.2 ([4, 2])**.** *For any C_{12}-manifold, we have*

$$\begin{aligned}R(X,Y)\xi&=-2\mathrm{d}\eta(X,Y)\psi-\eta(Y)\nabla_X\psi+\eta(X)\nabla_Y\psi, \qquad (2.10)\\ R(X,\xi)Y&=\omega(X)\big(\omega(Y)\xi-\eta(Y)\psi\big)+g(\nabla_X\psi,Y)\xi-\eta(Y)\nabla_X\psi,\\ S(X,\xi)&=-\eta(X)\operatorname{div}\psi,\end{aligned}$$

where R and S denote the Riemann curvature and the Ricci curvature tensors, respectively.

Example 2.3. We denote the Cartesian coordinates in a 3-dimensional Euclidean space $\mathbb{R}^3$ by (x_1, x_2, x_3) and define a metric tensor g by

$$g = e^{2f}\begin{pmatrix} \rho^2+\tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix},$$

where $f = f(y)$, $\tau = \tau(x)$ and $\rho = \rho(x, y)$ are functions on $\mathbb{R}^3$ with $f' = \frac{\partial f}{\partial y}$. Further, we define an almost contact structure (φ, ξ, η) on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\tau & 0 \end{pmatrix}, \qquad \xi = e^{-f}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = e^{f}\big(-\tau, 0, 1\big).$$

Thus

$$\mathrm{d}\eta = f'e^{f}\big(\tau \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z\big) \qquad \text{and} \qquad \mathrm{d}\phi = 0.$$

By direct computation, the non-zero components of $N_k^{(1)i}j$ are

$$N_{12}^{(1)3} = \tau f', \qquad N_{23}^{(1)3} = f'.$$

On the other hand,

$$(N_\varphi)^i_{kj} = 0 \qquad \text{for all } i, j, k \in \{1, 2, 3\}$$

implies that the structure (φ, ξ, η, g) is integrable. To ensure that the defined structure is not normal, it suffices to take $f' \neq 0$. Also, taking $\omega = f'\mathrm{d}y$, we can see that

$$\mathrm{d}\eta = \omega \wedge \eta, \quad \omega(\xi) = 0, \quad \text{and} \quad \mathrm{d}\omega = 0.$$

We denote by ψ the g-dual of ω,

$$\psi = \frac{f'}{\rho^2}e^{2f}\frac{\partial}{\partial y}.$$

Thus, $(M, \varphi, \xi, \psi, \eta, \omega, g)$ is a C_{12}-structure on $\mathbb{R}^3$.

A C_{12}-manifold M of dimension $2n+1$ with a C_{12}-structure (φ, ξ, η, g) is said to be η-Einstein if the Ricci curvature tensor S of the metric g satisfies the equation $S = \mu g + \nu \eta \otimes \eta$ for some constants $\mu, \nu \in \mathbb{R}$. In [10], Okumura assumed that both μ and ν are functions, and then proved, similarly to the case of Einstein metrics, that they must be constant when $n > 1$. Obviously, $\nu = 0$ reduces to the more familiar C_{12}-Einstein condition. In general, $\mu + \nu = -\operatorname{div}\psi$, and every C_{12}-η-Einstein manifold is necessarily of constant scalar curvature $r = 2n\mu - \operatorname{div}\psi$.

3. Lorentz C_{12}-manifolds

Contrary to the Riemann case, any smooth manifold cannot admit a Lorentz structure. In fact, this is possible if and only if there exists a global vector field (never vanishing) ([11, p. 149]). Additionally, the following well-known result shows how to obtain such a metric.

Proposition 3.1. *Let (M, g) be a Riemann manifold, V a unit global vector field and $V^\flat$ its dual 1-form. Then $\tilde{g} = g - 2V^\flat \otimes V^\flat$ is a Lorentz metric on M. Furthermore, V becomes timelike so the resulting Lorentz manifold is time orientable.*

Definition 3.2. Let $(M, \varphi, \xi, \eta, g)$ be a Lorentz almost contact manifold. M is said to be a *Lorentz almost contact C_{12}-manifold* if

$$\mathrm{d}\eta = \omega \wedge \eta, \qquad \mathrm{d}\phi = 0.$$

If, in addition, $N_\varphi = 0$, then $(M, \varphi, \xi, \eta, g)$ is called a *Lorentz C_{12}-manifold.*

Now, we consider a C_{12}-manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ and we obtain a Lorentz metric putting

$$\tilde{g} = g - 2\eta \otimes \eta. \tag{3.1}$$

Proposition 3.3. *The manifold $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ is a Lorentz C_{12}-manifold.*

Proof. Since (φ, ξ, η) is an almost contact structure, it is easy to see that ξ is timelike with respect to the metric $\tilde{g}$. We check the compatibility of $\tilde{g}$ with the structure; for any vector fields X and Y on M, we have

$$\begin{aligned}\tilde{g}(\varphi X, \varphi Y) &= g(\varphi X, \varphi Y)\\ &= g(X, Y) - \eta(X)\eta(Y)\\ &= \tilde{g}(X, Y) + \eta(X)\eta(Y).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\tilde{\phi}(X, Y) &= \tilde{g}(X, \varphi Y)\\ &= \phi(X, Y)\end{aligned}$$

for any vector fields X and Y on M. So we obtain $\mathrm{d}\tilde{\phi} = \mathrm{d}\phi = 0$.

Finally, the integrability condition (i.e., $N_\varphi = 0$) holds since it does not depend on the metric. Then, we obtain a Lorentz C_{12}-manifold on M. □

From now on, such a Lorentz C_{12}-manifold is said to be the associated Lorentz C_{12}-manifold.

Let $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ be a Lorentz C_{12}-manifold; considering the transformation

$$\overline{g} = \tilde{g} + 2\eta \otimes \eta,$$

one obtains that $(M^{2n+1}, \varphi, \xi, \eta, \overline{g})$ turns out to be a C_{12}-manifold.

To compare the Levi-Civita connections ∇ and $\tilde{\nabla}$ with respect to the Riemannian metric g and the Lorentz one $\tilde{g}$, we prove the following proposition.

Proposition 3.4. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a C_{12}-manifold and consider the associated Lorentz C_{12}-manifold $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$. Then, the Levi-Civita connections ∇ and $\tilde{\nabla}$ are related by*

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\eta(Y)\psi. \tag{3.2}$$

Proof. By Koszul's formula, for $\tilde{\nabla}$ we have

$$\begin{aligned}2\tilde{g}\big(\tilde{\nabla}_X Y, Z\big) &= X\tilde{g}(Y,Z) + Y\tilde{g}(Z,X) - Z\tilde{g}(X,Y)\\ &\quad + \tilde{g}(Z,[X,Y]) + \tilde{g}(Y,[Z,X]) - \tilde{g}(X,[Z,Y]),\end{aligned}$$

and, applying the definition of $\tilde{g}$, the above formula becomes

$$\begin{aligned}\tilde{g}\big(\tilde{\nabla}_X Y, Z\big) &= \tilde{g}\big(\nabla_X Y, Z\big) - \big(g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\big)\eta(Z)\\ &\quad - \mathrm{d}\eta(X,Z)\eta(Y) - \mathrm{d}\eta(Y,Z)\eta(X).\end{aligned}$$

With the help of (2.9) and (2.1), one can get

$$\tilde{g}\big(\tilde{\nabla}_X Y, Z\big) = \tilde{g}\big(\nabla_X Y, Z\big) + \eta(X)\eta(Y)\omega(Z),$$

and since

$$\begin{aligned}\omega(Z) &= g(\psi, Z)\\ &= \tilde{g}(\psi, Z),\end{aligned}$$

we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\eta(Y)\psi. \qquad \square$$

As a consequence of the relation between the Levi-Civita connections, we have the following theorem.

Theorem 3.5. *An almost contact Lorentzian manifold $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ is a Lorentz C_{12}-manifold if and only if*

$$(\tilde{\nabla}_X \varphi)Y = \eta(X)\omega(\varphi Y)\xi.$$

Proof. Using (3.2), we have

$$\begin{aligned}(\tilde{\nabla}_X \varphi)Y &= \tilde{\nabla}_X \varphi Y - \varphi\tilde{\nabla}_X Y\\ &= (\nabla_X \varphi)Y - \eta(X)\eta(Y)\varphi\psi;\end{aligned}$$

from (2.8), we get our formula. □

We now investigate some curvature properties of a Lorentz C_{12}-manifold.

Proposition 3.6. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a C_{12}-manifold and consider the associated Lorentz C_{12}-manifold $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$. Then, the curvature tensor R (resp., the Ricci curvature S) and the curvature tensor $\tilde{R}$ (resp., the Ricci curvature $\tilde{S}$) are related by*

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \eta(Z)R(X,Y)\xi, \tag{3.3}$$

$$\begin{aligned}\tilde{S}(X,Y) &= S(X,Y) + (2\,\mathrm{div}\,\psi - |\psi|^2)\eta(X)\eta(Y)\\ &\quad + \omega(X)\omega(Y) + g(\nabla_X \psi, Y)\end{aligned} \tag{3.4}$$

for all X, Y and Z vector fields on M.

Proof. Equation (3.3) follows from the formulas (3.2) and

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z,$$

with the help of (2.10). A standard orthonormalization process shows that if $\{e_i\}_{1\leq i\leq 2n+1}$ is a local orthonormal basis with respect to g, then it is a local pseudo-orthonormal basis with respect to $\tilde{g}$. Using the formula

$$\tilde{S}(X,Y) = \sum_{i=1}^{2n+1} \tilde{g}\big(\tilde{R}(X,e_i)e_i, Y\big),$$

by a simple computation using (3.3), (3.1) and (2.2), one can get the second relation. □

4. Ricci solitons in Lorentz C_{12}-manifolds

In [3], there are nice results on Ricci solitons on 3-dimensional C_{12}-manifolds; we confirm here that the results are valid for any odd dimension. In this section, we study the behavior of Ricci solitons and generalized Ricci solitons in the associated Lorentz C_{12}-manifold. We will start by introducing the basic concepts that we need in this section.

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a C_{12}-manifold and $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ its associated Lorentz C_{12}-manifold.

Proposition 4.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a C_{12}-manifold and consider the associated Lorentz C_{12}-manifold $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ admitting a (g, ξ, λ) Ricci soliton. The following holds:*

- *$(M, \tilde{g})$ is Einstein.*
- *The scalar curvature is given by $\tilde{r} = (2n+1)\operatorname{div}\psi$.*

Proof. Since $(\tilde{g}, \xi, \lambda)$ is a Ricci soliton, we have

$$\mathcal{L}_\xi \tilde{g}(X,Y) + 2\tilde{S}(X,Y) - 2\lambda \tilde{g}(X,Y) = 0.$$

From (3.2), we have

$$\tilde{\nabla}_X \xi = \nabla_X \xi + \eta(X)\psi,$$

and in view of (2.9), we get $\tilde{\nabla}_X \xi = 0$. Then, $\mathcal{L}_\xi \tilde{g}(X,Y) = 0$. Knowing that

$$\tilde{r} = \sum_{1}^{2n+1} \tilde{S}(e_i, e_i)$$

we obtain the desired result. □

For our first motivation, we consider the case where the potential field V is pointwise colinear with the vector field ξ, i.e., $V = f\xi$, where f is a function on M. We compute

$$\begin{aligned}(\mathcal{L}_{f\xi}\tilde{g})(X,Y) &= \tilde{g}\big(\tilde{\nabla}_X(f\xi), Y\big) + \tilde{g}\big(\tilde{\nabla}_Y(f\xi), X\big) \\ &= -X(f)\eta(Y) - Y(f)\eta(X).\end{aligned} \tag{4.1}$$

Replacing (4.1), (3.4) and (3.1) in (2.1), we obtain

$$\begin{aligned}(\mathcal{L}_{f\xi}\tilde{g})(X,Y)+2\tilde{S}(X,Y)-2\lambda\tilde{g}(X,Y)=\mathcal{L}_{\psi}g(X,Y)&+2\omega(X)\omega(Y)+2S(X,Y)\\&-2\lambda g(X,Y)-X(f)\eta(Y)-Y(f)\eta(X)\\&+2\big(2\lambda+2\operatorname{div}\psi-|\psi|^2\big)\eta(X)\eta(Y).\end{aligned}$$

$(\tilde{g},f\xi,\lambda)$ is a Ricci soliton if and only if

$$\begin{aligned}\mathcal{L}_{\psi}g(X,Y)=&-2\omega(X)\omega(Y)-2S(X,Y)+2\lambda g(X,Y)\\&+X(f)\eta(Y)+Y(f)\eta(X)-2\big(2\lambda+2\operatorname{div}\psi-|\psi|^2\big)\eta(X)\eta(Y).\end{aligned}\tag{4.2}$$

By setting $Y=\xi$ in (4.2), we obtain

$$X(f)=\big(2\lambda+2\operatorname{div}\psi-\xi(f)\big)\eta(X).\tag{4.3}$$

Again replacing X by ξ in (4.3), we get

$$\xi(f)=\lambda+\operatorname{div}\psi.$$

Substituting this in (4.3), we have

$$X(f)=\big(\lambda+\operatorname{div}\psi\big)\eta(X),\tag{4.4}$$

which implies

$$\mathrm{d}f=\big(\lambda+\operatorname{div}\psi\big)\eta.$$

Substituting (4.4) in (4.2), we obtain

$$\begin{aligned}\mathcal{L}_{\psi}g(X,Y)=&-2\omega(X)\omega(Y)-2S(X,Y)+2\lambda g(X,Y)\\&-2\big(\lambda+\operatorname{div}\psi-|\psi|^2\big)\eta(X)\eta(Y).\end{aligned}$$

Thus, we state the following:

Theorem 4.2. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a C_{12}-manifold and consider the associated Lorentz C_{12}-manifold $(M^{2n+1},\varphi,\xi,\eta,\tilde{g})$. If $(\tilde{g},f\xi,\lambda)$ is a Ricci soliton then g satisfies the generalized η-Ricci soliton equation* (2.4) *with $V=\psi$, $c_1=1$, $c_2=-1$, where $\mu=-2\big(\lambda+\operatorname{div}\psi-|\psi|^2\big)\in\mathbb{R}$ and $\mathrm{d}f=\big(\lambda+\operatorname{div}\psi\big)\eta$.*

In addition, if $\lambda=|\psi|^2-\operatorname{div}\psi\in\mathbb{R}$, then g satisfies the generalized Ricci soliton equation (2.2) *with $V=\psi$, $c_1=1$, $c_2=-1$, where $\mathrm{d}f=|\psi|^2\eta$.*

Conversely, suppose that g satisfies the generalized η-Ricci soliton equation (2.4) with $V=\psi$, that is,

$$\mathcal{L}_{\psi}g=-2c_1\omega\otimes\omega+2c_2S+2\lambda g+\mu\eta\otimes\eta,\tag{4.5}$$

where $c_1,c_2,\lambda,\mu\in\mathbb{R}$.

Using (3.4) and (3.1), taking into account $\mathcal{L}_{\psi}g(X,Y)=2g(\nabla_X\psi,Y)$, (4.5) reduces to

$$\begin{aligned}c_2\tilde{S}(X,Y)=-\lambda\tilde{g}(X,Y)&+\Big(c_2(2\operatorname{div}\psi-|\psi|^2)-2\lambda-\frac{\mu}{2}\Big)\eta(X)\eta(Y)\\&+(1+c_2)g(\nabla_X\psi,Y)+(c_1+c_2)\omega(X)\omega(Y),\end{aligned}\tag{4.6}$$

i.e., $\tilde{g}$ is η-Einstein if and only if

$$c_2=-1,\qquad c_1=1.$$

So (4.6) becomes

$$\tilde{S}(X,Y) = \lambda\tilde{g}(X,Y) + \Big(2\lambda + 2\operatorname{div}\psi - |\psi|^2 + \frac{\mu}{2}\Big)\,\eta(X)\eta(Y), \tag{4.7}$$

where $2\lambda + 2\operatorname{div}\psi - |\psi|^2 + \frac{\mu}{2} \in \mathbb{R}$. Furthermore, setting $X = Y = \xi$ in (4.7), we obtain

$$\frac{\mu}{2} = -\lambda - \operatorname{div}\psi + |\psi|^2,$$

and (4.7) reduces to

$$\tilde{S}(X,Y) = \lambda\tilde{g}(X,Y) - (\lambda + \operatorname{div}\psi)\eta(X)\eta(Y).$$

On the other hand, if g satisfies the generalized Ricci soliton equation (2.2) with $V = \psi$, from (4.7) with $\mu = 0$ we get

$$\tilde{S}(X,Y) = \lambda\tilde{g}(X,Y) + \big(2\lambda + 2\operatorname{div}\psi - |\psi|^2\big)\eta(X)\eta(Y), \tag{4.8}$$

where $2\lambda + 2\operatorname{div}\psi - |\psi|^2 \in \mathbb{R}$. Again, setting $X = Y = \xi$ in (4.8), we obtain

$$\lambda = |\psi|^2 - \operatorname{div}\psi,$$

and (4.8) reduces to

$$\tilde{S}(X,Y) = (|\psi|^2 - \operatorname{div}\psi)\tilde{g}(X,Y) + |\psi|^2\eta(X)\eta(Y).$$

Therefore, we have the following theorem.

Theorem 4.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a C_{12}-manifold with $\operatorname{div}\psi \in \mathbb{R}$ and consider the associated Lorentz C_{12}-manifold $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$.*

(1) *If g satisfies the generalized η-Ricci soliton equation (2.4) with*

$$V = \psi, \quad c_1 = 1, \quad c_2 = -1, \quad \frac{\mu}{2} = -\lambda - \operatorname{div}\psi + |\psi|^2,$$

then $(M, \tilde{g})$ is an η-Einstein manifold. In addition, if $\lambda = -\operatorname{div}\psi$, then $(M, \tilde{g})$ is an Einstein manifold.

(2) *If g satisfies the generalized Ricci soliton equation (2.2) with*

$$V = \psi, \quad c_1 = 1, \quad c_2 = -1, \quad \lambda = |\psi|^2 - \operatorname{div}\psi,$$

then $(M, \tilde{g})$ is an η-Einstein manifold.

For our second motivation, we consider the case where the potential field V is orthogonal to the Reeb vector field ξ.

Theorem 4.4. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a C_{12}-manifold, $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ the associated Lorentz C_{12}-manifold, and V a vector field on M orthogonal to ξ. If $(\tilde{g}, V, \lambda)$ is a Ricci soliton, then g satisfies the generalized η-Ricci bi-soliton equation (2.5) with*

$$V_1 = V + \psi, \quad V_2 = \psi, \quad c_1 = 1, \quad c_2 = -1, \quad \text{and} \quad \mu = |\psi|^2.$$

Proof. Let V be orthogonal to ξ, that is, $\eta(V)=0$. This implies

$$\begin{aligned}\eta(\nabla_X V) &= g(\nabla_X V,\xi)\\ &= -g(V,\nabla_X\xi)\\ &= \eta(X)\omega(V).\end{aligned}$$

So, using (3.2) and (3.1), one can get

$$\begin{aligned}\mathcal{L}_V\tilde{g}(X,Y) &= \tilde{g}\big(\tilde{\nabla}_X V,Y\big)+\tilde{g}\big(\tilde{\nabla}_Y V,X\big)\\ &= \mathcal{L}_V g(X,Y)-4\omega(V)\eta(X)\eta(Y).\end{aligned} \tag{4.9}$$

Then, from (4.9), (3.4) and (3.1) we obtain

$$\begin{aligned}\mathcal{L}_V\tilde{g}(X,Y)+2\tilde{S}(X,Y)-2\lambda\tilde{g}(X,Y) &= \mathcal{L}_V g(X,Y)+2S(X,Y)-2\lambda g(X,Y)\\ &\quad +2\omega(X)\omega(Y)+2g(\nabla_X\psi,Y)\\ &\quad +2\big(2\lambda+2\operatorname{div}\psi-2\omega(V)-|\psi|^2\big)\eta(X)\eta(Y).\end{aligned} \tag{4.10}$$

Suppose that $\mathcal{L}_V\tilde{g}(X,Y)+2\tilde{S}(X,Y)-2\lambda\tilde{g}(X,Y)=0$. Setting $X=Y=\xi$ we get

$$\lambda=\omega(V)-\operatorname{div}\psi.$$

Knowing that $\mathcal{L}_\psi g(X,Y)=2g(\nabla_X\psi,Y)$, the equation (4.10) becomes

$$\mathcal{L}_{(V+\psi)}g(X,Y)=-2\omega(X)\omega(Y)-2S(X,Y)+2\lambda g(X,Y)+2|\psi|^2\eta(X)\eta(Y).$$

This completes the proof. □

Now, suppose that (g,V,λ) is a Ricci soliton, that is, $\mathcal{L}_V g+2S-2\lambda g=0$ with V orthogonal to ξ. Setting $X=Y=\xi$, we obtain

$$\lambda=\omega(V)-\operatorname{div}\psi. \tag{4.11}$$

Using (4.1), (3.4), (3.1) and (4.11), one can get

$$\mathcal{L}_{(V-\psi)}\tilde{g}(X,Y)=2\omega(X)\omega(Y)-2\tilde{S}(X,Y)+2\lambda\tilde{g}(X,Y)+2|\psi|^2\eta(X)\eta(Y).$$

Therefore, we have the following theorem.

Theorem 4.5. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a C_{12}-manifold, $(M^{2n+1},\varphi,\xi,\eta,\tilde{g})$ the associated Lorentz C_{12}-manifold, and V a vector field on M orthogonal to ξ. If (g,V,λ) is a Ricci soliton, then $\tilde{g}$ satisfies the generalized η-Ricci bi-soliton equation* (2.5) *with*

$$V_1=V-\psi,\quad V_2=\psi,\quad c_1=-1,\quad c_2=1,\quad \text{and}\quad \mu=|\psi|^2.$$

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