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REMARKS ON A BOUNDARY VALUE PROBLEM FOR A MATRIX VALUED $\overline{\partial}$ EQUATION

CARLOS E. KENIG

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ABSTRACT. In this short note, we discuss a boundary value problem for a matrix valued $\overline{\partial}$ equation.

The problem we will discuss arose in [3], in the author's joint work with E. B. Davey and J.-N. Wang, on the Landis conjecture [6]. This conjecture states that if u is a real, bounded solution in $\mathbb{R}^N$ of $\Delta u = Vu$, where V is real, $\|V\|_\infty \leq 1$, and $|u(x)| \leq C_\epsilon \exp(-|x|^{1+\epsilon})$, $\epsilon > 0$ as $x \to \infty$, then $u \equiv 0$. In [5], the author, in joint work with L. Silvestre and J.-N. Wang observed that in the case when $N = 2$, and $V \geq 0$, one can use complex analysis to establish the conjecture (in quantitative form). In [3], with Davey and Wang, we showed the same result under a suitable (strong) decay assumption on V_-, the negative part of V. It was here that we were led to the matrix valued $\overline{\partial}$ equation that we discuss in this note. Afterwards, E. B. Davey, in [4], established the Landis conjecture under less strong decay on V_-, and finally, in [8], A. Logunov, E. Malinnikova, N. Nadirashvili, and F. Nazarov proved the full Landis conjecture when $N = 2$, also using complex methods. Let A be a 2×2 matrix with complex entries in $\mathbb{R}^2$, with $\|A\|_\infty \leq M$. Given H a bounded matrix on ∂D, where $D = \{z \in \mathbb{C} : |z| < 1\}$, assume that $\|H\|_\infty \leq N_1$, and that the matrix HH^* is strictly positive definite, with $\|(HH^*)^{-1}\|_\infty \leq N_2$. Consider the problem

$$\begin{cases} \overline{\partial} P = AP & \text{in } D \\ PP^* = HH^* & \text{in } \partial D, \text{ a.e. (non-tangentially)} \end{cases} \tag{1}$$

where P is a 2×2 complex matrix in D, and the boundary values are taken in the sense of non-tangential convergence.

Theorem 1. *There exists a solution P to (1), so that P and P^{-1} are bounded in $\overline{D}$. Moreover, if P_1, P_2 are two solutions, then $P_1 = P_2 U$, where U is a constant unitary matrix.*

Remark 2. Consider the scalar case of Theorem 1, namely when A and P are scalars. Let $\alpha(z) = \frac{1}{\pi} \int_{|\xi|<2} \frac{A(\xi)}{z-\xi}\, d\xi = T_{D_2}(A)(z)$, where T_{D_2} denotes the Cauchy–Pompeiu operator on the disc $D_2 = \{|\xi| < 2\}$. Then, $\overline{\partial}\alpha = A$ in D_2 and

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$|\alpha(z)| \lesssim M$ in $\overline{D}$. Let $q(z) = e^{-\alpha} P$, where P is as in (1). Then, $\overline{\partial} q = 0$ in D, $|q|^2 = e^{-2\operatorname{Re}\alpha}|P|^2 = e^{-2\operatorname{Re}\alpha}|H|^2$ on ∂D. The existence and uniqueness of q then follows from a classical theorem of Szegö [12], which in turn gives the existence and uniqueness of P (modulo unimodular constants for the uniqueness). Note that commutativity of the product is crucial for this argument.

Remark 3. Consider next the 2×2 matrix valued case, when $A \equiv 0$. Thus, P is a holomorphic matrix. Theorem 1 is then a consequence of the Wiener–Masani theorem [13, Theorem 7.13]. Note that the uniqueness assertion is not made in [13], but it is made and proved in [14]. More recent proofs of the Wiener–Masani theorem, under higher regularity assumptions and conclusions are given, for instance, in the works of Berndtsson–Rosay [1] and Lempert [7].

We now turn to the proof of Theorem 1. For the proof of the existence part of Theorem 1, we will combine the next Proposition 4, due to Davey–Kenig–Wang [3, Proposition 2] with the Wiener–Masani theorem.

Proposition 4. *Let A be a 2×2 matrix defined on $R = [-2, 2] \times [-2, 2]$, with $M = \|A\|_\infty$. There exists an invertible solution to $\overline{\partial} P_1 = A P_1$ in R, with the property that*

$$\|P_1\|_\infty + \|P_1^{-1}\|_\infty \lesssim \exp\left[CM^2 (\log M)^2\right].$$

Note that $\overline{\partial} P_1 = A P_1$, $\overline{\partial} P_1^{-1} = P_1^{-1} A$, and since the right-hand sides are bounded on R, P_1 and P_1^{-1} are in $C^\beta(\overline{D})$ $0 < \beta < 1$, with C^β norm bounded by $\exp[\tilde{C} M^2 (\log M)^2]$.

Proof of the existence part of Theorem 1. Let $\tilde{H} = P_1^{-1} H$, where P_1 is as in Proposition 4. Clearly, the invertibility of P_1^{-1} in R shows that, since HH^* is strictly positive and invertible on ∂D, so is $\tilde{H}\tilde{H}^*$. By the Wiener–Masani theorem (the case $A \equiv 0$ of Theorem 1), there exists Q invertible and bounded in $\overline{D}$, with Q^{-1} bounded, Q, Q^{-1} holomorphic in D, and $QQ^* = \tilde{H}\tilde{H}^*$ on ∂D. Let now $P = P_1 Q$. Since Q is holomorphic $\overline{\partial} P = AP$ in D. On ∂D, $PP^* = P_1 Q Q^* P_1^* = P_1 (P_1^{-1} H)(P_1^{-1} H)^* P_1^* = HH^*$, concluding the proof of existence.

For the proof of uniqueness, assume that $P, \tilde{P}$ are two solutions, as in Theorem 1. Let $Q = \tilde{P}^{-1} P$. Then $\overline{\partial} Q = \overline{\partial}(\tilde{P}^{-1}) P + \tilde{P}^{-1} \overline{\partial} P = -\tilde{P}^{-1} \overline{\partial} \tilde{P} \tilde{P}^{-1} P + \tilde{P}^{-1} A P = -\tilde{P}^{-1} A \tilde{P} \tilde{P}^{-1} P + \tilde{P}^{-1} A P = 0$. Also, on ∂D, $QQ^* = \tilde{P}^{-1} P P^* (\tilde{P}^{-1})^*$ $= \tilde{P}^{-1} HH^* (\tilde{P}^{-1})^*$. But $\tilde{P}\tilde{P}^* = HH^*$, so that $\tilde{P} = HH^*(\tilde{P}^*)^{-1}$, and $\tilde{P}^{-1} = \tilde{P}^*(HH^*)^{-1}$, hence $\tilde{P}^{-1} HH^* (\tilde{P}^{-1})^* = \tilde{P}^* (HH^*)^{-1} HH^* (\tilde{P}^{-1})^* = I$. Thus, $QQ^* = I$ on ∂D, and, $\overline{\partial} Q = 0$ in D. By the uniqueness in the Wiener–Masani theorem, $Q \equiv U$, U a constant unitary matrix, and so $P = \tilde{P} U$ as claimed. □

We next turn to a proof of the uniqueness in the Wiener–Masani theorem via the "multiplicative integral". The multiplicative integral is a multiplicative analog of the classical Riemann–Stieltjes integrals. It first arose in the work of V. Volterra (1887) on the study of systems of ordinary differential equations. See [2], [11], [9] for discussions of the topic. Here we follow the exposition in the Master's Thesis of Joris Roos (2014), which is unpublished, but can be found in [10]. The definition of the multiplicative integral that is given in [10] is a multiplicative analog of the Stieltjes one. We consider the space M_m or $m \times m$ matrices A, endowed with the matrix norm $\|A\| = \sup_{|x|=1} |Ax|$, where $|x| = (\sum_{j=1}^m x_j^2)^{1/2}$. We consider $t \in [a,b]$, and use the standard notion of a Hermitian matrix being positive, strictly positive, etc. We consider a partition $\tau = \{t_i\}_{i=0}^n$ of the interval $[a,b]$, $\Delta_i \tau = t_i - t_{i-1}$, $i = 1, \dots, n$, $\gamma(\tau) = \max_i \Delta_i \tau$. For a matrix valued function $E\colon [a,b] \to M_m$, we define $\operatorname{var}^\tau_{[a,b]} = \sum_{i=1}^n \|\Delta_i E\|$, where $\Delta_i E = \Delta_i^\tau E = E(t_i) - E(t_{i-1})$, and call E of bounded variation if $\operatorname{var}_{[a,b]} E = \sup_{\tau \in \tau_a^b} \operatorname{var}^\tau_{[a,b]} E < \infty$, $\tau_a^b = \{\text{all partitions of } [a,b]\}$. We denote by $\mathrm{BV}([a,b], M_m)$ the space of functions of bounded variation. We call $|E|(t) = \operatorname{var}_{[a,t]} E$. Given a partion τ, choose intermediate points $\xi = (\xi_i)_{i=1,\dots,n}$, $\xi_i \in [t_{i-1}, t_i]$. For f on $[a,b]$, with values in $\mathbb{C}$, or in M_m, we define $P(f, E, \tau, \xi) = P(\tau, \xi) = \prod_{i=1}^{\curvearrowright n} \exp(f(\xi_i)\Delta_i E)$. Here, $\prod_{i=1}^{\curvearrowright n} A_i = A_1 A_2 \cdots A_n$ denotes multiplication of the matrices $(A_i)_i$ from left to right. Let T_a^b be the set of tagged partitions (τ, ξ), such that τ is a subdivision of $[a,b]$ and ξ is a choice of corresponding intermediate points. We say that $P \in M_m$ is the (right) multiplicative Stieltjes integral corresponding to $f\colon [a,b] \to M_m$ (or $\mathbb{C}$), $E\colon [a,b] \to M_m$, if $\forall \epsilon > 0$, there exists a $(\tau_0, \xi_0) \in T_a^b$ such that $\|P(\tau,\xi) - P\| < \epsilon$ for every $(\tau, \xi) < (\tau_0, \xi_0)$, i.e. for all $\tau \subset \tau_0$. One can show that if $f\colon [a,b] \to \mathbb{C}$ is continuous and $E\colon [a,b] \to M_m$ is of bounded variation, then $\int_a^{\curvearrowright b} \exp(f dE)$, which is, by definition, the right multiplicative integral just defined, exists.

An important result (see [10, Proposition 2.7]) is

$$\det \int_a^{\curvearrowright b} \exp\left(f(t)\, \mathrm{d}E(t)\right) = \exp\left(\int_a^b f(t)\, \mathrm{d}\operatorname{tr} E(t)\right), \tag{2}$$

where $\operatorname{tr} A$ is the trace of the matrix A. Note that, in particular, multiplicative integrals always yield invertible matrices.

Next we sketch a proof of the uniqueness in the Wiener–Masani theorem, using multiplicative integrals. Thus, let $\overline{\partial} Q = 0$ in D, Q, Q^{-1} bounded in $\overline{D}$, $QQ^* = I$ on ∂D. Note first that $\|Q(z)\| \le 1$ for all $z \in D$, since $\|Q\|$ is subharmonic [10, Lemma A.4], $\|Q(z)\| = 1$, $z \in \partial D$. Then, by [10, Theorem 3.1], Potapov's decomposition [9], $Q(z) = B(z) \int_0^{\curvearrowright L} \exp(h_z(\theta(t))\, \mathrm{d}E(t))$, where $B(z)$ is a Blaschke–Potapov product corresponding to the zeros of $\det Q$, $0 \le L \le \infty$, E is an increasing matrix valued function such that $\operatorname{tr} E(t) = t$, $t \in [0, L]$, $\theta\colon [0,L] \to [0, 2\pi]$ is a right continuous increasing function, and $h_z(\theta) = \frac{z + e^{\mathrm{i}\theta}}{z - e^{\mathrm{i}\theta}}$ is the Herglotz kernel. Since $\det Q(z) \neq 0$ in $\overline{D}$, $B(z) = U$, U a constant unitary matrix.

Next we claim that $q(z) = \det Q(z) \equiv e^{\mathrm{i}\theta_0}$ in $\overline{D}$. Assuming this, we have that

$$\begin{aligned} 1 = |e^{\mathrm{i}\theta_0}| = |q(0)| &= |\det Q(0)| \\ = \left|\det U \int_0^L \exp(h_0(\theta(t))\, \mathrm{d}E(t))\right| &= \left|\det \int_0^L \exp(h_0(\theta(t))\, \mathrm{d}E(t))\right| \end{aligned}$$

$$(\text{by } (2)) = \left|\exp\int_0^L h_0(\theta(t))\,\mathrm{d}\operatorname{tr}E(t)\right| = \left|\exp\int_0^L \frac{e^{\mathrm{i}\theta(t)}}{-e^{\mathrm{i}\theta(t)}}\,\mathrm{d}t\right| = \exp(-L).$$

But then, $L = 0$, $Q(z) = U$.

We turn to the proof of the claim. Let $q(z) \neq 0$, $z \in D$, q, q^{-1} be holomorphic in D, bounded in $\overline{D}$, $|q(z)| = 1$, $z \in \partial D$. Consider $u(z) = \log|q(z)|$, which is harmonic in D, bounded on $\overline{D}$ (since q^{-1} is bounded in $\overline{D}$). Then $u(0) = \log|q(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|q(e^{\mathrm{i}\theta})|\,\mathrm{d}\theta = 0$. Also, since $u(z) \equiv 0$, $z \in \partial D$, by the maximum principle $u(z) \leq 0$ in D. But since $u(0) = 0$, $u \equiv 0$, so $|q(z)| \equiv 1$, $z \in D$, and since q is holomorphic in D, q is constant, and thus $q(z) \equiv e^{\mathrm{i}\theta_0}$.

Finally, we turn to the main open question, which motivates this note. Let H in Theorem 1 be the identity matrix, and P the corresponding solution. By the construction of P and Proposition 4, we know that $\|P\|_\infty$ and $\|P^{-1}\|_\infty$ are bounded by $\exp(CM^2(\log M)^2)$, where $\|A\|_\infty \leq M$, and we assume, for convenience, that $M \geq 1$. We would like to know:

Question 5. *Are $\|P\|_\infty$, $\|P^{-1}\|_\infty$ bounded by $\exp(C_\epsilon M^{1+\epsilon})$, for each $\epsilon > 0$?*

An affirmative answer to this question would give, following the argument in [3], a proof of the Landis conjecture for $N = 2$ (which is now the theorem of Logunov–Malinnikova–Nadirashvili–Nazarov [8]). Notice that we can reduce ourselves to the case when $\operatorname{tr} A = 0$, and hence $\det P \equiv e^{\mathrm{i}\theta_0}$, so that $\|P\|_\infty = \|P^{-1}\|_\infty$. Indeed, a simple computation yields that, if $q = \det P$, then $\overline{\partial} q = (\operatorname{tr} A) q$, so that, if $\operatorname{tr} A = 0$, q is holomorphic, and so is q^{-1}, and $|q(z)| = 1$, $z \in \partial D$, since on ∂D, $PP^* = I$. Thus as before, $q(z) = e^{\mathrm{i}\theta_0}$. To reduce to the $\operatorname{tr} A = 0$ case, note that if $A = A_1 + A_2$, $\overline{\partial} P_1 = A_1 P_1$, and P_2 solves $\overline{\partial} P_2 = P_2 B$, where $B = -P_1^{-1} A_2 P_1$, then $P = P_1 P_2^{-1}$ solves $\overline{\partial} P = AP$. Let $A_2 = \begin{pmatrix} \frac{\operatorname{tr} A}{2} & 0 \\ 0 & \frac{\operatorname{tr} A}{2} \end{pmatrix}$, $A_1 = A - A_2$, so that $\operatorname{tr}(A_1) = 0$. Also, since A_2 is a scalar matrix, $B = -P_1^{-1} A_2 P_1 = -A_2$, so that $\overline{\partial} P_2 = P_2 A_2 = A_2 P_2$. Since A_2 is scalar, and for the scalar equation we have the exponential bounds with M to the power 1 (see Remark 2), and $P = P_1 P_2^{-1}$, it suffices to give the bounds for P_1, which solves $\overline{\partial} P_1 = A_1 P_1$, with $\operatorname{tr} A_1 = 0$. (In the case of the Landis conjecture, the matrix in [3] has trace 0 to begin with). This is a challenging question in its own right.

Final remark. It was with great sadness that I learned of the unexpected death of Pola Harboure. Pola and I became good friends during the time that she spent at Minnesota in the early 1980s and we kept in touch over the years. Her death is a great loss for mathematics, especially in Argentina and Latin America, where she was a pillar of the mathematical community. It is also a great loss for her family and friends, for whom she was so important. We continue to mourn her.

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Carlos E. Kenig
Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
ckenig@uchicago.edu

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