REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 66, No. 1, 2023, Pages 1–34
Published online: September 21, 2023
https://doi.org/10.33044/revuma.4327

VARIATION AND OSCILLATION OPERATORS ON WEIGHTED MORREY–CAMPANATO SPACES IN THE SCHRÖDINGER SETTING

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Dedicated to the memory of our friend and colleague Eleonor Harboure

Abstract. We denote by $\mathcal{L}$ the Schrödinger operator with potential V, that is, $\mathcal{L} = -\Delta + V$, where it is assumed that V satisfies a reverse Hölder inequality. We consider weighted Morrey–Campanato spaces $\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)$ and $\mathrm{BLO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)$ in the Schrödinger setting. We prove that the variation operator $V_\sigma(\{T_t\}_{t>0})$, $\sigma > 2$, and the oscillation operator $O(\{T_t\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$, where $t_j < t_{j+1}$, $j \in \mathbb{Z}$, $\lim\limits_{j\to+\infty} t_j = +\infty$ and $\lim\limits_{j\to-\infty} t_j = 0$, being $T_t = t^k \partial_t^k e^{-t\mathcal{L}}$, $t > 0$, with $k \in \mathbb{N}$, are bounded operators from $\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)$ into $\mathrm{BLO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)$. We also establish the same property for the maximal operators defined by $\{t^k \partial_t^k e^{-t\mathcal{L}}\}_{t>0}$, $k \in \mathbb{N}$.

1. Introduction

Let $\{T_t\}_{t>0}$ be a family of bounded operators in $L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Many times we are interested in knowing the behavior of T_t when $t \to 0^+$. Specifically we want to know if there exists the limit $\lim\limits_{t\to 0^+} T_t(f)(x)$ for almost everywhere $x \in \mathbb{R}^d$ when $f \in L^p(\mathbb{R}^d)$. A first way to deal with the problem is to consider the maximal operator T_* defined by $T_* f = \sup\limits_{t>0} |T_t f|$. If T_* defines a bounded operator from $L^p(\mathbb{R}^d)$ into $L^{p,\infty}(\mathbb{R}^d)$ and $\lim\limits_{t\to 0^+} T_t(g)(x)$ exists for almost all $x \in \mathbb{R}^d$ when $g \in \mathcal{D}$, where $\mathcal{D}$ is a dense subspace of $L^p(\mathbb{R}^d)$, then $\lim\limits_{t\to 0^+} T_t(t)(x)$ exists for almost all $x \in \mathbb{R}^d$ when $f \in L^p(\mathbb{R}^d)$. This procedure is well known and it is named Banach principle ([23, pp. 27–28]). Another approach to studying this question is based on the variation operator. Let $\sigma > 2$. The variation operator

2020 *Mathematics Subject Classification.* 43A85, 42B20, 42B25.

Key words and phrases. Variation operator, oscillation operator, Morrey–Campanato spaces, Schrödinger operator.

The authors are partially supported by grant PID2019-106093GB-I00 from the Spanish Government.

$V_\sigma(\{T_t\}_{t>0})$ is defined by

$$V_\sigma(\{T_t\}_{t>0})(f)(x) = \sup_{\substack{0<t_n<t_{n-1}<\cdots<t_1 \\ n\in\mathbb{N}}} \left(\sum_{j=1}^{n-1} \left| T_{t_{j+1}}(f)(x) - T_{t_j}(f)(x) \right|^\sigma \right)^{\frac{1}{\sigma}}.$$

If $V_\sigma(\{T_t\}_{t>0})(f)(x) < \infty$, then there exists the limit $\lim_{t\to 0^+} T_t(f)(x)$.

We observe that in this case it is not necessary to have the existence of the limit when f is in a dense subset of $L^p(\mathbb{R}^d)$. In order to see the measurability of $V_\sigma(\{T_t\}_{t>0})(f)$ when $f \in L^p(\mathbb{R}^d)$ we need additional properties for $\{T_t\}_{t>0}$. For instance, if for almost all $x \in \mathbb{R}^d$ the function $t \longrightarrow T_t(f)(x)$ is continuous in $(0, \infty)$, then

$$V_\sigma(\{T_t\}_{t>0})(f)(x) = \sup_{\substack{0<t_n<t_{n-1}<\ldots<t_1 \\ t_j\in\mathbb{Q}, j=1,\ldots,n \\ n\in\mathbb{N}}} \left(\sum_{j=1}^{n-1} \left| T_{t_{j+1}}(f)(x) - T_{t_j}(f)(x) \right|^\sigma \right)^{\frac{1}{\sigma}}$$

for almost all $x \in \mathbb{R}^d$ and $V_\sigma(\{T_t\}_{t>0}(f)$ is measurable in $\mathbb{R}^d$. Once the measurability property is assumed to be of interest to studying the boundedness of the variation operators in function spaces. Note that if $V_\sigma(\{T_t\}_{t>0})(f)(x)$ defines a bounded operator in L^p, BMO, Lipschitz or Hardy spaces, for instance, then $V_\sigma(\{T_t\}_{t>0})(f)(x) < \infty$ for almost all $x \in \mathbb{R}^d$, when f belongs to those function spaces. Furthermore, the boundedness properties of the variation operator inform us about the speed of convergence of $T_t(f)(x)$ as $t \to 0^+$.

Variational inequalities have been extensively studied in the last two decades in probability, ergodic theory, and harmonic analysis. Lépingle ([33]) established the first variational inequality involving martingales improving the classical Doob maximal inequality. Bourgain ([17]), some years later, proved a variational inequality for the ergodic average of a dynamic system. Since then many authors have studied variation operators in harmonic analysis (see, for instance, [1], [20], [21], [22], [30], [36], [37], [38], [39] and [40]).

In order to obtain L^p-variation inequalities it is usual to need $\sigma > 2$ (see [20, Remark 1.7] and [41]). When $\sigma = 2$ a good substitute is the oscillation operator defined as follows. Suppose that $\{t_j\}_{j\in\mathbb{Z}}$ is a sequence of positive numbers such that $0 < t_j < t_{j+1} < \infty$, $j \in \mathbb{Z}$, $\lim_{j\to-\infty} t_j = 0$ and $\lim_{j\to+\infty} t_j = +\infty$. We define the oscillation operator associated with $\{t_j\}_{j\in\mathbb{Z}}$ for $\{T_t\}_{t>0}$ by

$$O(\{T_t\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(x) = \left(\sum_{j\in\mathbb{Z}} \sup_{t_j \le \varepsilon_j < \varepsilon_{j+1} < t_{j+1}} \left| T_{\varepsilon_j}(f)(x) - T_{\varepsilon_{j+1}}(f)(x) \right|^2 \right)^{\frac{1}{2}}.$$

Note that if the exponent 2 in the last definition is replaced by another one greater than 2, the new operator is controlled by that with exponent 2.

Finally we recall the definition of the short variation operator $SV(\{T_t\}_{t>0})$. For every $k \in \mathbb{Z}$ we define

$$V_k(\{T_t\}_{t>0})(f)(x) = \sup_{\substack{2^{-k}<t_n<\ldots<t_1\leq 2^{-k+1} \\ n\in\mathbb{N}}} \left(\sum_{j=1}^{n-1} |T_{t_j}(f)(x) - T_{t_{j+1}}(f)(x)|^2 \right)^{\frac{1}{2}}.$$

The short variation operator $SV(\{T_t\}_{t>0})$ is given by

$$SV(\{T_t\}_{t>0})(f)(x) = \left(\sum_{k\in\mathbb{Z}} (V_k(\{T_t\}_{t>0})(f)(x))^2 \right)^{\frac{1}{2}}.$$

Our objective in this paper is to study the variation, oscillation, and short variation operators when $T_t = t^k \partial_t^k S_t$, $t > 0$, with $k \in \mathbb{N}$, where $\{S_t\}_{t>0}$ represents the heat or Poisson semigroup associated with the Schrödinger operator in $\mathbb{R}^d$. We consider weighted Morrey–Campanato spaces in the Schrödinger setting.

We denote by $\mathcal{L}$ the Schrödinger operator in $\mathbb{R}^d$, $d \geq 3$, defined by

$$\mathcal{L} = -\Delta + V,$$

where $\Delta = \sum_{i=1}^{d} \partial_{x_i}^2$ represents the Euclidean Laplacian and the potential $V \geq 0$ is not identically zero and it belongs to q-reverse Hölder class (in short, $V \in RH_q(\mathbb{R}^d)$), that is, there exists $C > 0$ such that

$$\left(\frac{1}{|B|} \int_B V(x)^q \, dx \right)^{\frac{1}{q}} \leq \frac{C}{B} \int_B V(x) \, dx$$

for every ball B in $\mathbb{R}^d$. The class $RH_q(\mathbb{R}^d)$, is defined in this way for $1 < q < \infty$. Every nonnegative polynomial is in $RH_q(\mathbb{R}^d)$ for each $1 < q < \infty$.

Harmonic analysis associated with the operator $\mathcal{L}$ has been developed by several authors in the last century. Shen's paper [42] can be considered the starting point of the most of these studies (see, for instance, [24], [25], [26], [31], [35], [43] and [48]). Professor Eleonor Harboure, to whose memory this paper is dedicated, studied several important aspects of the harmonic analysis in the Schrödinger setting ([2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [18] and [27]).

The following function ρ, which is named *critical radius*, plays an important role and it is defined by

$$\rho(x) = \sup \left\{ r \in (0, \infty) : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y) \, dy \leq 1 \right\}.$$

The Schrödinger operator $\mathcal{L}$ becomes a nice perturbation of the Euclidean Laplacian, which means that the harmonic analysis operators (Riesz transforms, multipliers, Littlewood–Paley functions) have the same behaviour close to the diagonal as the corresponding Euclidean operators. The closeness to the diagonal is defined by the critical radius function, The main properties of the function ρ were established in [42, Lemma 1.4].

By a weight w we understand a measurable and positive function in $\mathbb{R}^d$. As in [14] we say that a weight w is in $A_p^{\rho,\theta}(\mathbb{R}^d)$, with $1 < p < \infty$ and $\theta > 0$, when there exists $C > 0$ such that, for every ball B in $\mathbb{R}^d$,

$$\left(\frac{1}{\Psi_\theta(B)|B|}\int_B w(y)dy\right)\left(\frac{1}{\Psi_\theta(B)|B|}\int_B w^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1} \leq C.$$

Here, if $x \in \mathbb{R}^d$ and $r > 0$,

$$\Psi_\theta(B(x,r)) = \left(1 + \frac{r}{\rho(x)}\right)^\theta.$$

We define $A_p^{\rho,\infty}(\mathbb{R}^d) = \cup_{\theta>0} A_p^{\rho,\theta}(\mathbb{R}^d)$, $1 < p < \infty$.

In [14] and [44] the main properties of the weights in $A_p^{\rho,\infty}(\mathbb{R}^d)$ were proved.

We now define the Morrey–Campanato spaces $\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ and $\mathrm{BLO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$.

Let $w \in A_p^{\rho,\infty}(\mathbb{R}^d)$ and $\alpha \in [0,1)$. A locally integrable function f on $\mathbb{R}^d$ is said to be in $\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ when there exists $C > 0$ such that, for every $x_0 \in \mathbb{R}^d$ and $0 < r_0 < \rho(x_0)$,

$$\frac{1}{|B(x_0,r_0)|^\alpha w(B(x_0,r_0))}\int_{B(x_0,r_0)} \left|f(y) - f_{B(x_0,r_0)}\right| dy \leq C, \tag{1.1}$$

where

$$f_{B(x_0,r_0)} = \frac{1}{|B(x_0,r_0)|}\int_{B(x_0,r_0)} f(y)\,dy, \quad \text{and } r_0 > 0,$$

and, when $x_0 \in \mathbb{R}^d$ and $r_0 \geq \rho(x_0)$,

$$\frac{1}{|B(x_0,r_0)|^\alpha w(B(x_0,r_0))}\int_{B(x_0,r_0)} |f(y)|\,dy \leq C. \tag{1.2}$$

We define

$$\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} = \inf\left\{C > 0 : (1.1) \text{ and } (1.2) \text{ hold}\right\}.$$

As it is proved in [45, Lemma 2.1] in (1.2) it is sufficient to consider $r_0 = \rho(x_0)$.

We say that a function $f \in \mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ is in $\mathrm{BLO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ when there exists $C > 0$ such that, for each $x_0 \in \mathbb{R}^d$ and $0 < r_0 < \rho(x_0)$,

$$\frac{1}{|B(x_0,r_0)|^\alpha w(B(x_0,r_0))}\int_{B(x_0,r_0)} \left(f(y) - \operatorname*{ess\,inf}_{z\in B(x_0,r_0)} f(z)\right) dy \leq C. \tag{1.3}$$

We define

$$\|f\|_{\mathrm{BLO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} = \inf\left\{C > 0 : (1.2) \text{ and } (1.3) \text{ hold}\right\}.$$

It is clear that $\mathrm{BLO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ is contained in $\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$.

Note that the spaces $\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ and $\mathrm{BLO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ actually depend on the critical radius function ρ but here we prefer to point out the dependence of the operator $\mathcal{L}$.

The operator $-\mathcal{L}$ generates a semigroup of operators $\{W_t^{\mathcal{L}} := e^{-t\mathcal{L}}\}_{t>0}$ on $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, where, for every $t > 0$,

$$W_t^{\mathcal{L}}(f)(x) = \int_{\mathbb{R}^d} W_t^{\mathcal{L}}(x,y) f(y)\, dy, \quad f \in L^p(\mathbb{R}^d),\ 1 \le p < \infty.$$

$\{W_t^{\mathcal{L}}\}_{t>0}$ is also named the heat semigroup associated with $\mathcal{L}$. For every $t > 0$, the kernel $W_t^{\mathcal{L}}(\cdot,\cdot)$ is a positive symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ and satisfies that $\int_{\mathbb{R}^d} W_t^{\mathcal{L}}(x,y)\, dy \le 1$. The semigroup $\{W_t^{\mathcal{L}}\}_{t>0}$ is not Markovian.

By using subordination formula ([52, pp. 259–268]), for every $\beta \in (0,1)$, the semigroup of operators $\{W_{\beta,t}^{\mathcal{L}}\}_{t>0}$ generated by $-\mathcal{L}^\beta$ is defined by

$$W_{\beta,t}^{\mathcal{L}}(f) = \int_0^\infty \eta_t^\beta(s) W_s^{\mathcal{L}}(f) ds, \quad t > 0,$$

where η_t^β is a certain nonnegative continuous function. The special case $\{W_{1/2,t}^{\mathcal{L}}\}_{t>0}$ is known as Poisson semigroup associated with $\mathcal{L}$.

In [25, Theorem 6] it was proved that the maximal operators $W_*^{\mathcal{L}}$ and $W_{1/2,*}^{\mathcal{L}}$ defined by

$$W_*^{\mathcal{L}}(f) = \sup_{t>0} |W_t^{\mathcal{L}}(f)| \quad \text{and} \quad W_{1/2,*}^{\mathcal{L}}(f) = \sup_{t>0} |W_{1/2,t}^{\mathcal{L}}(f)|,$$

are bounded from $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$ into itself, where by $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$ we represent the space $\mathrm{BMO}_{\mathcal{L},w}^\alpha(\mathbb{R}^d)$ when $w = 1$ and $\alpha = 0$. Ma, Stinga, Torrea, and Zhang ([35, Theorem 1.3]) proved that $W_*^{\mathcal{L}}$ and $W_{1/2,*}^{\mathcal{L}}$ are bounded from $\mathrm{BMO}_{\mathcal{L}}^\alpha(\mathbb{R}^d)$ into itself, where $\mathrm{BMO}_{\mathcal{L}}^\alpha(\mathbb{R}^d)$ denotes the space $\mathrm{BMO}_{\mathcal{L},w}^\alpha(\mathbb{R}^d)$ with $w = 1$. In [51, Proposition 5.2, (i)] it was established that $W_*^{\mathcal{L}}$ and $W_{1/2,*}^{\mathcal{L}}$ are bounded from $E_\rho^{\alpha,p}(\mathbb{R}^d)$ into $\widetilde{E}_\rho^{\alpha,p}(\mathbb{R}^d)$, when $1 < p < \infty$, and where these spaces are defined like $\mathrm{BMO}_{\mathcal{L},w}^\alpha(\mathbb{R}^d)$ and $\mathrm{BLO}_{\mathcal{L},w}^\alpha(\mathbb{R}^d)$, but where the L^1-norm is replaced by the L^p-norm and $w = 1$.

We now consider, for every $k \in \mathbb{N}$, the maximal operators

$$W_*^{\mathcal{L},k}(f) = \sup_{t>0} |t^k \partial_t^k W_t^{\mathcal{L}}(f)|.$$

Our first result is the following.

Theorem 1.1. *Let $k \in \mathbb{N}$, $q > d/2$, and $\alpha \in [0,1)$. Suppose that $V \in RH_q(\mathbb{R}^d)$ and that $w \in A_p^{\rho,\theta}(\mathbb{R}^d)$ for some $\theta > 0$ such that $2(d(p+\alpha-1)+p\theta) < \min\{1, 2 - d/q\}$. Then, the maximal operators $W_*^{\mathcal{L},k}$ are bounded from $\mathrm{BMO}_{\mathcal{L},w}^\alpha(\mathbb{R}^d)$ into $\mathrm{BLO}_{\mathcal{L},w}^\alpha(\mathbb{R}^d)$.*

The variation operator $V_\sigma(\{W_t^{\mathcal{L}}\}_{t>0})$ was studied in [2] and [3]. In [3, Theorem 2.6] it was proved that $V_\sigma(\{W_t^{\mathcal{L}}\}_{t>0}$ is bounded from $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$ into itself. This result was extended by Bui ([19]) when the Schrödinger operator $\mathcal{L}$ is replaced by another operator L such that the kernel of e^{-tL}, $t > 0$, satisfies the same properties as the kernel of $e^{-t\mathcal{L}}$ (see [19, p. 125]). Tang and Zhang ([45]) generalized [3, Theorem 2.6] proving that $V_\sigma(\{W_t^{\mathcal{L}}\}_{t>0})$ is bounded from $\mathrm{BMO}_{\mathcal{L},w}^\alpha(\mathbb{R}^d)$ into itself (see [45, Theorem 5]). We extend this last property as follows. The theorem is a complement of the results given in [53].

Theorem 1.2. *Let $k \in \mathbb{N}$, $q > d/2$, $\alpha \in [0,1)$, $\sigma > 2$, and $1 < p < \infty$. Suppose that $V \in RH_q(\mathbb{R}^d)$ and that $w \in A_p^{\rho,\theta}(\mathbb{R}^d)$, for some $\theta > 0$, and $\{t_j\}_{j\in\mathbb{Z}}$ is a sequence of positive numbers satisfying that $t_j < t_{j+1}$, $j \in \mathbb{Z}$, $\lim_{j\to+\infty} t_j = +\infty$, $\lim_{j\to-\infty} t_j = 0$. If $2(d(p+\alpha-1)+p\theta) < \min\{1, 2-d/q\}$, then the operators $V_\sigma\left(\left\{t^k\partial_t^k W_t^{\mathcal{L}}\right\}_{t>0}\right)$, $O\left(\left\{t^k\partial_t^k W_t^{\mathcal{L}}\right\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}}\right)$ and $SV\left(\left\{t^k\partial_t^k W_t^{\mathcal{L}}\right\}_{t>0}\right)$ are bounded from $\mathrm{BMO}_{\mathcal{L},w}^{\alpha}(\mathbb{R}^d)$ into $\mathrm{BLO}_{\mathcal{L},w}^{\alpha}(\mathbb{R}^d)$.*

In the proof of Theorems 1.1 and 1.2 we are inspired by the ideas developed by Da. Yang, Do. Yang, and Zhou ([49], [50] and [51]) and Tang and Zhang ([45]).

We organize the paper as follows. In section 2 we recall some properties about the kernels, the weights, and the spaces that will be useful in the proofs of our results. The proof of Theorem 1.2 for the variation operator is given in section 3. We prove Theorem 1.2 for the oscillation operator in section 4. In section 5 we give a proof of Theorem 1.2 for the short variation operator. A sketch of the proof of Theorem 1.1 is presented in section 6.

Our arguments allow us also to prove the same properties when the semigroup $\{W_t^{\mathcal{L}}\}_{t>0}$ is replaced by $\{W_{\beta,t}^{\mathcal{L}}\}_{t>0}$, with $\beta \in (0,1)$. We also remark that the methods we have used can be applied to establish versions of Theorems 1.1 and 1.2 when the operator $\mathcal{L}$ is replaced by the following ones:

(a) Generalized Schrödinger operators defined by $\mathfrak{L} = -\Delta + \mu$ on $\mathbb{R}^d$, where μ is a nonnegative Radon measure on $\mathbb{R}^d$ satisfying certain scale-invariant Kato condition ([43] and [48]).

(b) Degenerate Schrödinger operators on $\mathbb{R}^d$ defined as follows. Let w belong to the Muckenhoupt class $A_2(\mathbb{R}^d)$ and let $\{a_{ij}\}_{i,j=1}^d$ be a real symmetric matrix function satisfying that

$$\frac{1}{C}|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq C|\xi|^2, \quad x, \xi \in \mathbb{R}^d.$$

The degenerate Schrödinger operator is defined by

$$\mathfrak{L}(f)(x) = -\frac{1}{w(x)}\sum_{i,j=1}^{d} \partial_i(a_{ij}(\cdot)\partial_j f)(x) + V(x).$$

Here V satisfies certain integrability conditions with respect to the measure $w(x)\,dx$ ([27]).

(c) Schrödinger operators on $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_n$ defined by $\mathfrak{L} = -\Delta_{\mathbb{H}^n} + V$, where $\Delta_{\mathbb{H}^n}$ represents the sublaplacian in $\mathbb{H}^n$ ([34]).

(d) Schrödinger operators on connected and simply connected nilpotent Lie groups G defined by $\mathfrak{L} = -\Delta_G + V$, where Δ_G denotes the sublaplacian in G ([46]).

Throughout this paper by c and C we always denote positive constants that can change in each occurrence.

2. Some auxiliary results

In this section we present some results that will be useful in what follows. We begin with some properites of the Schrödinger heat kernel.

Proposition 2.1. *Let $k \in \mathbb{N}$ and $q > d/2$.*

(a) *For every $N \in \mathbb{N}$, there exists $C = C(N)$ such that*

$$\left|t^k \partial_t^k W_t^{\mathcal{L}}(x,y)\right| \leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \quad x, y \in \mathbb{R}^d \text{ and } t > 0.$$

(b) *For every $0 < \delta < \min\{1, 2 - d/q\}$ and $N \in \mathbb{N}$, there exists $C = C(N, \delta)$ such that, for every $x, y, h \in \mathbb{R}^d$, $t > 0$ and $|h| \leq \sqrt{t}$,*

$$\left|t^k \partial_t^k W_t^{\mathcal{L}}(x+h,y) - t^k \partial_t^k W_t^{\mathcal{L}}(x,y)\right| \leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

(c) *For every $0 < \delta \leq \min\{1, 2 - d/q\}$ and $N \in \mathbb{N}$, there exists $C = C(N, \delta)$ such that*

$$\left|\int_{\mathbb{R}^d} t^k \partial_t^k W_t^{\mathcal{L}}(x,y)\, dy\right| \leq C \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N}, \quad x \in \mathbb{R}^d \text{ and } t > 0.$$

(d) *There exists $C > 0$ such that, for each $x, y \in \mathbb{R}^d$ and $t > 0$,*

$$\left|t^k \partial_t^k W_t^{\mathcal{L}}(x,y) - t^k \partial_t^k W_t(x-y)\right| \leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \left(\frac{\sqrt{t}}{\max\{\rho(x), \rho(y)\}}\right)^{2-\frac{d}{q}}.$$

Here, W_t represents the classical heat kernel.

Proof. The properties (a), (b) and (c) were proved in [28, Proposition 3.3]. The property (d) was established in [47, Proposition 1]. □

In what follows we denote $\delta_0 := \min\{1, 2 - d/q\}$.

We now list the main properties of the weights in $A_p^{\rho,\theta}(\mathbb{R}^d)$.

Proposition 2.2 ([44, Lemma 2.2], [45, Proposition 2.4])**.** *Let $1 < p < \infty$ and $\theta > 0$.*

(a) $w \in A_p^{\rho,\theta}(\mathbb{R}^d)$ if and only if $w^{-\frac{1}{p-1}} \in A_{p'}^{\rho,\theta}(\mathbb{R}^d)$, where $p' = \frac{p}{p-1}$.

(b) If $w \in A_p^{\rho,\theta}(\mathbb{R}^d)$, there exists $C > 0$ such that

$$\frac{w(B)}{w(E)} \leq C \left(\frac{\psi_\theta(B)|B|}{|E|}\right)^p,$$

for every ball B in $\mathbb{R}^d$ and every measurable set $E \subset B$.

(c) If $w \in A_p^{\rho,\theta}(\mathbb{R}^d)$ for every $c \geq 1$, there exists $C > 0$ such that

$$\frac{w(2^k B)}{w(B)} \leq C 2^{kp(\theta+d)}$$

for every $k \in \mathbb{Z}$ and every ball $B = B(x,r)$ being $r \leq c\rho(x)$.

Concerning Morrey–Campanato spaces $\mathrm{BMO}_{\mathcal{L},w}^{\alpha}(\mathbb{R}^d)$ we will use the following result.

Proposition 2.3 ([45, Corollary 2.1]). *Let* $1 < p < \infty$, $\theta > 0$, $\alpha \in [0, 1)$, $\nu \in (1, p']$, *and* $w \in A_p^{\rho,\theta}(\mathbb{R}^d)$. *For every* $c \geq 1$, *there exist* $C > 0$ *such that, if* $f \in \mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$, *then*

$$\frac{1}{|B|^\alpha}\left(\frac{1}{w(B)}\int_B |f(y) - f_B|^\nu w(y)^{1-\nu} dy\right)^{1/\nu} \leq C\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} \tag{2.1}$$

for every $B = B(x, r)$ *being* $0 < r \leq c\rho(x)$, *and, for a certain* $\gamma > 0$,

$$\frac{1}{|B|^\alpha}\left(\frac{1}{w(B)}\int_B |f(y)|^\nu w(y)^{1-\nu} dy\right)^{1/\nu} \leq C\left(1 + \frac{r}{\rho(x)}\right)^\gamma \|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} \tag{2.2}$$

for every $B = B(x, r)$ *with* $r \geq \rho(x)$.

3. Proof of Theorem 1.2 for the variation operator $V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})$

We have to see that there exists $C > 0$ such that, for every $f \in \mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$,

(i) for every $x_0 \in \mathbb{R}^d$,

$$\int_B \left|V_\sigma\left(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}\right)(f)(x)\right| dx \leq C|B|^\alpha w(B)\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)},$$

where $B = B(x_0, \rho(x_0))$;

(ii) for each $x_0 \in \mathbb{R}^d$ and $0 < r < \rho(x_0)$,

$$\int_B \left(V_\sigma\left(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}\right)(f)(x) - \alpha(B, f)\right) dx \leq C|B|^\alpha w(B)\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)},$$

where $\alpha(B, f) = \operatorname{ess\,inf}_{y\in B} V_\sigma\left(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}\right)(f)(y)$ and $B = B(x_0, r)$.

In [45, Theorem 4] it was proved the variation operator $V_\sigma(\{W_t^{\mathcal{L}}\}_{t>0})$ is bounded from $L^p(\mathbb{R}^d, w)$ into itself. By Proposition 2.1 the k-th derivative $\partial_t^k W_t^{\mathcal{L}}(x, y)$ of the heat kernel satisfies all the properties that we need to establish, by proceeding as in the proof of [45, Theorem 4], that the variation operator $V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})$ is bounded from $L^p(\mathbb{R}^d, w)$ into itself. Then, by using Proposition 2.1, (a), as in the proof of [45, Theorem 5, p. 610], we can see that the property (i) holds.

We are going to prove (ii). Let $f \in \mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and $0 < r_0 < \rho(x_0)$. We take $0 < t_n < t_{n-1} < \ldots < t_1$. In the case that $t_{i_0+1} < 8r_0^2 \leq t_{i_0}$ for some $i_0 \in \{1, \ldots, n-1\}$, by understanding the sums in the suitable way when $i_0 = n-1$, the Minkowski inequality implies that, for every $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\left(\sum_{i=1}^{n-1}\left|t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i+1}} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_i}\right|^\sigma\right)^{1/\sigma}\\
&= \Bigg[\left(\sum_{i=1}^{i_0-1} + \sum_{i=i_0+1}^{n-1}\right)|t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i+1}} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_i}|^\sigma\\
&\qquad + \Big|(t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i_0+1}} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=8r_0^2})\\
&\qquad + (t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=8r_0^2} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i_0}})\Big|^\sigma\Bigg]^{1/\sigma}
\end{aligned}$$

$$\leq \Big[\sum_{i=1}^{i_0-1} |t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i+1}} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_i}|^\sigma$$

$$+ |t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=8r_0^2} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i_0}}\Big|^\sigma \Big]^{1/\sigma}$$

$$+ \Big[\sum_{i=i_0+1}^{n-1} |t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i+1}} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_i}|^\sigma$$

$$+ \Big|(t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i_0+1}} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=8r_0^2})\Big|^\sigma \Big]^{1/\sigma},$$

and, if $8r_0^2 \leq t_n$, we can write

$$\Big(\sum_{i=1}^{n-1} |t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i+1}} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_i}|^\sigma \Big)^{1/\sigma}$$

$$\leq \Big(|t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=8r_0^2} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_n}|^\sigma$$

$$+ \sum_{i=1}^{n-1} |t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{i+1}} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_i}|^\sigma \Big)^{1/\sigma}.$$

We thus deduce that

$$V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f) \leq V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t\in(0,8r_0^2]})(f) + V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t\in[8r_0^2,\infty)})(f).$$

On the other hand, it is clear that

$$V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f) \geq V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t\in[8r_0^2,\infty)})(f).$$

Also we have that

$$V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t\in[8r_0^2,\infty)})(f)(x) - \operatorname*{ess\,inf}_{y\in B(x_0,r_0)} V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(y)$$

$$\leq V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t\in[8r_0^2,\infty)})(f)(x) - \operatorname*{ess\,inf}_{y\in B(x_0,r_0)} V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t\in[8r_0^2,\infty)})(f)(y)$$

$$\leq \operatorname*{ess\,sup}_{z,y\in B(x_0,r_0)} \Big|V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t\in[8r_0^2,\infty)})(f)(z) - V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t\in[8r_0^2,\infty)})(f)(y)\Big|$$

$$\leq \operatorname*{ess\,sup}_{z,y\in B(x_0,r_0)} \sup_{8r_0^2\leq t_n<\ldots<t_1} \Big(\sum_{i=1}^{n-1} \big|(t^k\partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=t_i} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=t_{i+1}})$$

$$- (t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=t_i} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=t_{i+1}})\big|^\sigma \Big)^{1/\sigma}, \quad \text{a.e. } x\in B(x_0,r_0).$$

It follows that

$$\int_{B(x_0,r_0)} \Big(V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(x) - \operatorname*{ess\,inf}_{y\in B(x_0,r_0)} V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(y) \Big)\, dx$$

$$\leq \int_{B(x_0,r_0)} V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t\in(0,8r_0^2]})(f)(x)\, dx + |B(x_0,r_0)|$$

$$\times \operatorname*{ess\,sup}_{z,y\in B(x_0,r_0)} \sup_{8r_0^2\le t_n<\ldots<t_1} \Big(\sum_{i=1}^{n-1} \big| [t^k\partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=t_i} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=t_{i+1}}]$$
$$- [t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=t_i} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=t_{i+1}}]\big|^\sigma \Big)^{1/\sigma}$$
$$=: G_1(f) + G_2(f).$$

We now estimate $G_1(f)$ and $G_2(f)$ separately. Firstly we consider $G_1(f)$. The function f is decomposed as follows:

$$f = (f - f_{B(x_0,r_0)})\mathcal{X}_{B(x_0,2r_0)} + (f - f_{B(x_0,r_0)})\mathcal{X}_{B(x_0,2r_0)^c} + f_{B(x_0,r_0)} =: f_1 + f_2 + f_3.$$

It is clear that $G_1(f) \le \sum_{j=1}^3 G_1(f_j)$. Also, by Proposition 2.2 (a), $w^{-\frac{1}{p-1}} \in A_{p'}^{\rho,\theta}(\mathbb{R}^d)$. Then, $V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})$ is bounded from $L^{p'}(\mathbb{R}^d, w^{-\frac{1}{p-1}})$ into itself. It follows that

$$\begin{aligned}
G_1(f_1) &\le w(B(x_0,r_0))^{1/p} \left(\int_{\mathbb{R}^d} |V_\sigma(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f_1)(x)|^{p'} w^{-\frac{1}{p-1}}(x)\,dx \right)^{1/p'} \\
&\le Cw(B(x_0,r_0))^{1/p} \left(\int_{B(x_0,2r_0)} |f(x) - f_{B(x_0,r_0)}|^{p'} w^{-\frac{1}{p-1}}(x)\,dx \right)^{1/p'} \\
&\le Cw(B(x_0,r_0))^{1/p} \left(\Big(\int_{B(x_0,2r_0)} |f(x) - f_{B(x_0,2r_0)}|^{p'} w^{-\frac{1}{p-1}}(x)\,dx \Big)^{1/p'} \right. \\
&\qquad \left. + |f_{B(x_0,2r_0)} - f_{B(x_0,r_0)}| \Big(\int_{B(x_0,2r_0)} w(x)^{-\frac{1}{p-1}}\,dx \Big)^{1/p'} \right) \\
&\le Cw(B(x_0,r_0))^{1/p} \Big(w(B(x_0,2r_0))^{1/p'} |B(x_0,2r_0)|^\alpha \|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} \qquad (3.1) \\
&\qquad + \frac{1}{w(B(x_0,2r_0))^{1/p}} \int_{B(x_0,2r_0)} |f(x) - f_{B(x_0,2r_0)}|\,dx \Big) \\
&\le Cw(B(x_0,r_0))^{1/p} w(B(x_0,2r_0)^{1/p'} |B(x_0,2r_0)|^\alpha \|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} \\
&\le C|B(x_0,r_0)|^\alpha w(B(x_0,r_0)) \|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)}. \qquad (3.2)
\end{aligned}$$

In (3.1) we use estimate (2.1) and that $w \in A_p^{\rho,\theta}(\mathbb{R}^d)$. In (3.2) we have taken into account Proposition 2.2 (c).

To analyze $G_1(f_2)$ we write

$$\begin{aligned}
G_1(f_2) &= \int_{B(x_0,r_0)} \sup_{0<t_n<\ldots<t_1\le 8r_0^2} \left(\sum_{i=1}^{n-1} \Big| \int_{t_{i+1}}^{t_i} \partial_t(t^k\partial_t^k W_t^{\mathcal{L}}(f_2)(x))\,dt \Big|^\sigma \right)^{1/\sigma} dx \\
&\le \int_{B(x_0,r_0)} \int_0^{8r_0^2} \big| \partial_t(t^k\partial_t^k W_t^{\mathcal{L}}(f_2)(x)) \big|\,dtdx.
\end{aligned}$$

According to Proposition 2.1, (a), we have that

$$\begin{aligned}
G_1(f_2) &\leq C\int_{B(x_0,r_0)}\int_{\mathbb{R}^d\setminus B(x_0,2r_0)}|f(y)-f_{B(x_0,r_0)}|\int_0^{8r_0^2} e^{-c\frac{||^2}{t}}t^{-\frac{d}{2}-1}\,dtdydx\\
&\leq C\int_{B(x_0,r_0)}\int_{\mathbb{R}^d\setminus B(x_0,2r_0)}|f(y)-f_{B(x_0,r_0)}|\frac{e^{-c\frac{|x-y|^2}{r_0^2}}}{|x-y|^d}\,dydx\\
&\leq C\int_{B(x_0,r_0)}\int_{\mathbb{R}^d\setminus B(x_0,2r_0)}|f(y)-f_{B(x_0,r_0)}|\frac{e^{-c\frac{|x_0-y|^2}{r_0^2}}}{|x_0-y|^d}\,dydx\\
&\leq C|B(x_0,r_0)|\sum_{i=1}^{\infty}\frac{e^{-c2^{2i}}}{(2^ir_0)^d}\int_{B(x_0,2^{i+1}r_0)\setminus B(x_0,2^ir_0)}|f(y)-f_{B(x_0,r_0)}|\,dy\\
&\leq C\sum_{i=1}^{\infty}\frac{e^{-c2^{2i}}}{2^{id}}\int_{B(x_0,2^{i+1}r_0)}|f(y)-f_{B(x_0,r_0)}|\,dy\\
&\leq C\sum_{i=1}^{\infty}\frac{e^{-c2^{2i}}}{2^{id}}\Big(\int_{B(x_0,2^{i+1}r_0)}|f(y)-f_{B(x_0,2^{i+1}r_0)}|\,dy\\
&\quad+|B(x_0,2^{i+1}r_0)|\sum_{j=0}^{i}|f_{B(x_0,2^{j+1}r_0)}-f_{B(x_0,2^jr_0)}|\Big).
\end{aligned}$$

We now observe that, for every $n\in\mathbb{N}$, according to Proposition 2.2 (c),

$$\begin{aligned}
\int_{B(x_0,2^nr_0)}|f(y)-f_{B(x_0,2^nr_0)}|\,dy &\leq C|B(x_0,2^nr_0)|^{\alpha}w(B(x_0,2^nr_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}\\
&\leq C2^{n(d(p+\alpha)+p\theta)}|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)},
\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}
|f_{B(x_0,2^{n+1}r_0)}-f_{B(x_0,2^nr_0)}| &\leq \frac{1}{|B(x_0,2^nr_0)|}\int_{B(x_0,2^{n+1}r_0)}|f(y)-f_{B(x_0,2^{n+1}r_0)}|\,dy\\
&\leq C2^{n(d(p+\alpha-1)+p\theta)}|B(x_0,r_0)|^{\alpha-1}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.
\end{aligned}\tag{3.4}$$

Thus,

$$\begin{aligned}
G_1(f_2) &\leq C|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}\\
&\qquad\times\sum_{i=1}^{\infty}e^{-c2^{2i}}\Big(2^{i(d(p+\alpha-1)+p\theta)}+\sum_{j=0}^{i}2^{j(d(p+\alpha-1)+p\theta)}\Big)\\
&\leq C|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.
\end{aligned}$$

Let us deal now with $G_1(f_3)$. By $\{W_t\}_{t>0}$ we denote the classical heat semigroup, that is, for every $t>0$,

$$W_t(f)=\int_{\mathbb{R}^d}W_t(x-y)f(y)\,dy,\quad x\in\mathbb{R}^d,$$

where

$$W_t(z) = \frac{1}{(4\pi t)^{d/2}} e^{-|z|^2/4t}, \;\; z \in \mathbb{R}^d.$$

By taking into account that $\partial_t(t^k \partial_t^k W_t(1)(x)) = 0$, $x \in \mathbb{R}^d$ and $t > 0$, we can write

$$\begin{aligned}
G_1(f_3) &\leq |f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \int_0^{8r_0^2} \Big| \int_{\mathbb{R}^d} \partial_t(t^k \partial_t^k W_t^{\mathcal{L}}(x,y))\, dy \Big| \, dtdx \\
&= |f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \int_0^{8r_0^2} \Big| \int_{\mathbb{R}^d} \partial_t(t^k \partial_t^k W_t^{\mathcal{L}}(x,y) - t^k \partial_t^k W_t(x-y))\, dy \Big| \, dtdx \\
&\leq |f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \int_0^{8r_0^2} \int_{|x-y|\leq \rho(x_0)} |\partial_t(t^k \partial_t^k W_t^{\mathcal{L}}(x,y) - t^k \partial_t^k W_t(x-y))| \, dydtdx \\
&\quad + |f_{B(x_0,r_0)}| \\
&\qquad \times \int_{B(x_0,r_0)} \int_0^{8r_0^2} \int_{|x-y|\geq \rho(x_0)} |\partial_t(t^k \partial_t^k W_t^{\mathcal{L}}(x,y) - t^k \partial_t^k W_t(x-y))| \, dydtdx \\
&=: G_{11}(f_3) + G_{12}(f_3).
\end{aligned}$$

According to Proposition 2.1(d) we have that

$$\begin{aligned}
G_{11}(f_3) &\leq C|f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \int_0^{8r_0^2} \int_{|x-y|\leq \rho(x_0)} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{2-\frac{d}{q}} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}+1}} \, dydtdx \\
&\leq C|f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \int_{|x-y|\leq \rho(x_0)} \frac{e^{-c\frac{|x-y|^2}{r_0^2}}}{\rho(x)^{2-\frac{d}{q}}} \int_0^{8r_0^2} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}+\frac{d}{2q}}} \, dtdydx \\
&\leq C|f_{B(x_0,r_0)}| \rho(x_0)^{\frac{d}{q}-2} \int_{B(x_0,r_0)} \int_{|x-y|\leq \rho(x_0)} \frac{e^{-c\frac{|x-y|^2}{r_0^2}}}{|x-y|^{d+\frac{d}{q}-2}} \, dydx \\
&\leq C|f_{B(x_0,r_0)}| \rho(x_0)^{\frac{d}{q}-2} |B(x_0,r_0)| \int_0^{\rho(x_0)} e^{-c\frac{s^2}{r_0^2}} s^{1-\frac{d}{q}} ds \\
&\leq C|f_{B(x_0,r_0)}| \rho(x_0)^{\frac{d}{q}-2} |B(x_0,r_0)| \int_0^{\infty} e^{-c\frac{s^2}{r_0^2}} s^{1-\frac{d}{q}} ds \\
&= C|f_{B(x_0,r_0)}| |B(x_0,r_0)| \left(\frac{r_0}{\rho(x_0)}\right)^{2-\frac{d}{q}}.
\end{aligned}$$

In the third inequality we have taken into account that $\rho(x) \sim \rho(x_0)$ provided that $|x - x_0| < \rho(x_0)$.

On the other hand, by Proposition 2.1, (a), and since

$$|t^k \partial_t^k W_t(z)| \leq \frac{C}{t^{d/2}} e^{-c|z|^2/t}, \;\; z \in \mathbb{R}^d \text{ and } t > 0, \tag{3.5}$$

we have that

$$\begin{aligned} G_{12}(f_3) &\leq C|f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \int_0^{8r_0^2} \int_{|x-y|\geq\rho(x_0)} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}+1}} \, dydtdx \\ &\leq C|f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \int_{|x-y|\geq\rho(x_0)} e^{-c\frac{|x-y|^2}{r_0^2}} \int_0^{8r_0^2} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}+1}} \, dtdydx \\ &\leq C|f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \int_{|x-y|\geq\rho(x_0)} \frac{e^{-c\frac{|x-y|^2}{r_0^2}}}{|x-y|^d} \, dydx \\ &\leq C|f_{B(x_0,r_0)}||B(x_0,r_0)| \int_{\rho(x_0)}^{\infty} \frac{e^{-c\frac{s^2}{r_0^2}}}{s} \, ds \\ &\leq C|f_{B(x_0,r_0)}||B(x_0,r_0)| \left(\frac{r_0}{\rho(x_0)}\right)^{\beta}, \end{aligned}$$

provided that $\beta > 0$.

We deduce that, for $\beta > 0$,

$$G_1(f_3) \leq C|f_{B(x_0,r_0)}||B(x_0,r_0)| \left(\left(\frac{r_0}{\rho(x_0)}\right)^{2-\frac{d}{q}} + \left(\frac{r_0}{\rho(x_0)}\right)^{\beta} \right). \tag{3.6}$$

We now choose $i_0 \in \mathbb{N}$ such that $2^{i_0} r_0 < \rho(x_0) \leq 2^{i_0+1} r_0$. By (3.4) we get

$$\begin{aligned} |f_{B(x_0,r_0)}| &\leq \sum_{i=0}^{i_0} |f_{B(x_0,2^{i+1}r_0)} - f_{B(x_0,2^i r_0)}| + |f_{B(x_0,2^{i_0+1}r_0)}| \\ &\leq C|B(x_0,r_0)|^{\alpha-1} w(B(x_0,r_0)) \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)} \sum_{i=0}^{i_0+1} 2^{i(d(p+\alpha-1)+p\theta)} \\ &\leq C 2^{i_0(d(p+\alpha-1)+p\theta)} |B(x_0,r_0)|^{\alpha-1} w(B(x_0,r_0)) \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)} \\ &\leq C \left(\frac{\rho(x_0)}{r_0}\right)^{d(p+\alpha-1)+p\theta} |B(x_0,r_0)|^{\alpha-1} w(B(x_0,r_0)) \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}. \end{aligned} \tag{3.7}$$

Since $2 - \frac{d}{q} > d(p+\alpha-1)+p\theta$ and taking $\beta = d(p+\alpha-1)+p\theta$ in (3.6) we obtain

$$G_1(f_3) \leq C|B(x_0,r_0)|^{\alpha} w(B(x_0,r_0)) \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.$$

By putting together the above estimations we obtain

$$G_1(f) \leq C|B(x_0,r_0)|^{\alpha} w(B(x_0,r_0)) \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}. \tag{3.8}$$

We now deal with $G_2(f)$. We can write

$$\begin{aligned}G_2(f) &\leq |B(x_0,r_0)| \operatorname*{ess\ sup}_{x,y\in B(x_0,r_0)} \int_{8r_0^2}^\infty \Big|\int_{\mathbb{R}^d} \partial_t(t^k\partial_t^k W_t^{\mathcal{L}}(x,z) - t^k\partial_t^k W_t^{\mathcal{L}}(y,z))f(z)dz\Big| dt\\
&\leq |B(x_0,r_0)| \operatorname*{ess\ sup}_{x,y\in B(x_0,r_0)} \left(\int_{8\rho(x_0)^2}^\infty + \int_{8r_0^2}^{8\rho(x_0)^2}\right)\\
&\quad \times \Big|\int_{\mathbb{R}^d} \partial_t(t^k\partial_t^k W_t^{\mathcal{L}}(x,z) - t^k\partial_t^k W_t^{\mathcal{L}}(y,z))f(z)dz\Big| dt\\
&=: G_{21}(f) + G_{22}(f).\end{aligned}$$

We firstly estimate $G_{21}(f)$. According to Proposition 2.1, (b), we deduce that, for every $0<\delta<\delta_0$, there exists $C>0$ such that, for each $x,y\in B(x_0,r_0)$ and $t>\rho(x_0)^2$,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d} \partial_t(t^k\partial_t^k W_t^{\mathcal{L}}(x,z) - t^k\partial_t^k W_t^{\mathcal{L}}(y,z))f(z)\,dz\Big|\\
&\leq C\int_{\mathbb{R}^d}\left(\frac{|x-y|}{\sqrt{t}}\right)^\delta \frac{e^{-c\frac{|y-z|^2}{t}}}{t^{\frac{d}{2}+1}}|f(z)|\,dz\\
&\leq \frac{C}{t^{\frac{d}{2}+1}}\left(\frac{|x-y|}{\sqrt{t}}\right)^\delta\left(\sum_{j=0}^\infty e^{-c2^{2j}}\int_{2^{j-1}\sqrt{t}\leq|y-z|<2^j\sqrt{t}}|f(z)|\,dz + \int_{|y-z|<2^{-1}\sqrt{t}}|f(z)|\,dz\right)\\
&\leq \frac{C}{t^{\frac{d}{2}+1}}\left(\frac{|x-y|}{\sqrt{t}}\right)^\delta\left(\sum_{j=0}^\infty e^{-c2^{2j}}\int_{|x_0-z|<2^{j+1}\sqrt{t}}|f(z)|\,dz + \int_{|x_0-z|<\sqrt{t}}|f(z)|\,dz\right)\\
&\leq \frac{C}{t^{\frac{d}{2}+1}}\left(\frac{r_0}{\sqrt{t}}\right)^\delta \|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)}\sum_{j=0}^\infty e^{-c2^{2j}}|B(x_0,2^j\sqrt{t})|^\alpha w(B(x_0,2^j\sqrt{t}))\\
&\leq \frac{C}{t}\left(\frac{r_0}{\sqrt{t}}\right)^\delta \|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} w(B(x_0,r_0))\frac{t^{\frac{d}{2}(p+\alpha-1)+\frac{p\theta}{2}}}{r_0^{p(\theta+d)}}.\end{aligned}$$

In the last inequality we have used Proposition 2.2 (b) and (c). For every $x,y\in B(x_0,r_0)$ and $t>8\rho(x_0)^2$,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d} \partial_t(t^k\partial_t^k [W_t^{\mathcal{L}}(x,z) - t^k W_t^{\mathcal{L}}(y,z)])f(z)dz\Big|\\
&\quad\leq C\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} w(B(x_0,r_0))|B(x_0,r_0)|^{\alpha-1} r_0^{\delta-d(p+\alpha-1)-p\theta} t^{\frac{d}{2}(p+\alpha-1)-\frac{\delta-p\theta}{2}-1}.\end{aligned}$$

It follows that

$$\begin{aligned}G_{21}(f) &\leq C\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} w(B(x_0,r_0))|B(x_0,r_0)|^\alpha\\
&\qquad\times r_0^{\delta-d(p+\alpha-1)-p\theta}\int_{8\rho(x_0)^2}^\infty t^{\frac{d}{2}(p+\alpha-1)-\frac{\delta-p\theta}{2}-1}\,dt\\
&\leq C\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} w(B(x_0,r_0))|B(x_0,r_0)|^\alpha\left(\frac{r_0}{\rho(x_0)}\right)^{\delta-d(p+\alpha-1)-p\theta}\\
&\leq C\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)} w(B(x_0,r_0))|B(x_0,r_0)|^\alpha,\end{aligned}$$

provided that $\delta > d(p+\alpha-1)+p\theta$. Note that we can choose this δ because $\delta_0 > d(p+\alpha-1)+p\theta$.

To deal with $G_{22}(f)$ we write, for every $t \in (8r_0^2, 8\rho(x_0)^2)$ and $x, y \in B(x_0, r_0)$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} \partial_t(t^k\partial_t^k[W_t^{\mathcal{L}}(x,z)-W_t^{\mathcal{L}}(y,z)])f(z)dz\Big|\\
&\quad\le \Big|\int_{\mathbb{R}^d} \partial_t(t^k\partial_t^k[W_t^{\mathcal{L}}(x,z)-W_t^{\mathcal{L}}(y,z)])(f(z)-f_{B(x_0,r_0)})dz\Big|\\
&\qquad+|f_{B(x_0,r_0)}|\Big|\int_{\mathbb{R}^d} \partial_t(t^k\partial_t^k[W_t^{\mathcal{L}}(x,z)-W_t^{\mathcal{L}}(y,z)])dz\Big| =: F_1(x,y,t)+F_2(x,y,t).
\end{aligned}$$

Thus,

$$G_{22}(f) \le C|B(x_0,r_0)| \sup_{x,y\in B(x_0,r_0)} \int_{8r_0^2}^{8\rho(x_0)^2} (F_1(x,y,t)+F_2(x,y,t))\,dt.$$

By using again Proposition 2.1, (b), for every $0<\delta<\delta_0$, and proceeding as above we have that, for every $x, y \in B(x_0, r_0)$ and $8r_0^2 < t < 8\rho(x_0)^2$,

$$\begin{aligned}
F_1(x,y,t) &\le C\int_{\mathbb{R}^d}\left(\frac{|x-y|}{\sqrt{t}}\right)^{\delta}\frac{e^{-c\frac{|y-z|^2}{t}}}{t^{\frac{d}{2}+1}}|f(z)-f_{B(x_0,r_0)}|dz\\
&\le \frac{C}{t^{\frac{d}{2}+1}}\left(\frac{r_0}{\sqrt{t}}\right)^{\delta}\left(\sum_{j=0}^{\infty} e^{-c\frac{2^{2j}r_0^2}{t}}\int_{2^jr_0\le|y-z|<2^{j+1}r_0}|f(z)-f_{B(x_0,r_0)}|\,dz\right.\\
&\qquad\left.+\int_{|y-z|<2^{-1}r_0}|f(z)-f_{B(x_0,r_0)}|\,dz\right)\\
&\le \frac{C}{t^{\frac{d}{2}+1}}\left(\frac{r_0}{\sqrt{t}}\right)^{\delta}\left(\sum_{j=0}^{\infty} e^{-c\frac{2^{2j}r_0^2}{t}}\int_{B(x_0,2^{j+1}r_0)}|f(z)-f_{B(x_0,r_0)}|\,dz\right.\\
&\qquad\left.+\int_{B(x_0,r_0)}|f(z)-f_{B(x_0,r_0)}|\,dz\right)\\
&\le \frac{C}{t^{\frac{d}{2}+1}}\left(\frac{r_0}{\sqrt{t}}\right)^{\delta}\left(\sum_{j=0}^{\infty} e^{-c\frac{2^{2j}r_0^2}{t}}\left[\int_{B(x_0,2^{j+1}r_0)}|f(z)-f_{B(x_0,2^{j+1}r_0)}|\,dz\right.\right.\\
&\qquad\left.+|B(x_0,2^{j+1}r_0)|\sum_{i=0}^{j}|f_{B(x_0,2^{i+1}r_0)}-f_{B(x_0,2^ir_0)}|\right]\\
&\qquad\left.+\int_{B(x_0,r_0)}|f(z)-f_{B(x_0,r_0)}|\,dz\right).
\end{aligned}$$

Now, according to (3.3) and (3.4) we obtain, for every $x,y \in B(x_0,r_0)$ and $8r_0^2 < t < 8\rho(x_0)^2$,

$$F_1(x,y,t) \leq \frac{C}{t^{\frac{d}{2}+1}}\left(\frac{r_0}{\sqrt{t}}\right)^{\delta}|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0)\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)} \times\left(\sum_{j=0}^{\infty} e^{-c\frac{2^{2j}r_0^2}{t}}(j+1)2^{j(d(p+\alpha)+p\theta)}+1\right).$$

It follows that

$$\begin{aligned}\int_{8r_0^2}^{8\rho(x_0)^2} F_1(x,y,t)\,dt &\leq Cr_0^{\delta}|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)} \\ &\quad\times\left(\sum_{j=0}^{\infty}(j+1)2^{j(d(p+\alpha)+p\theta)}\int_{8r_0^2}^{8\rho(x_0)^2}\frac{e^{-c\frac{2^{2j}r_0^2}{t}}}{t^{\frac{d+\delta}{2}+1}}\,dt+\int_{8r_0^2}^{8\rho(x_0)^2}\frac{dt}{t^{\frac{d+\delta}{2}+1}}\right) \\ &\leq Cr_0^{\delta}|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)} \\ &\quad\times\left(\sum_{j=0}^{\infty}\frac{(j+1)2^{j(d(p+\alpha)+p\theta)}}{(2^j r_0)^{d+\delta}}+\frac{1}{r_0^{d+\delta}}\right) \\ &\leq C|B(x_0,r_0)|^{\alpha-1}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}, \quad x,y\in B(x_0,r_0),\end{aligned}$$

provided that $\delta > d(p+\alpha-1)+p\theta$.

Finally, let $m \in \mathbb{N}$. By Proposition 2.1, (c), there exists $C>0$ such that

$$\left|\int_{\mathbb{R}^d} t^m\partial_t^{m+1}W_t^{\mathcal{L}}(x,y)\,dy\right| \leq \frac{C}{t}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}, \quad t\leq 8\rho(x)^2 \text{ and } x\in\mathbb{R}^d,$$

and by, [51, p. 98], for every $0<\delta<\delta_0$, there exists $C>0$ such that, for every $x,y \in B(x_0,r_0)$ and $t > 8r_0^2$,

$$|t^m\partial_t^{m+1}(W_t^{\mathcal{L}}(1)(x)-W_t^{\mathcal{L}}(1)(y))| \leq \frac{C}{t}\left(\frac{r_0}{\sqrt{t}}\right)^{\delta}.$$

By using these estimates we have, for every $x,y\in B(x_0,r_0)$ and $t\in\left[8r_0^2, 8\rho(x_0)^2\right]$,

$$\begin{aligned}F_2(x,y,t) &\leq C|f_{B(x_0,r_0)}|\sum_{m=k-1}^{k}|t^m\partial_t^{m+1}(W_t^{\mathcal{L}}(1)(x)-W_t^{\mathcal{L}}(1)(y))|^{1/2} \\ &\quad\times\left[\left|\int_{\mathbb{R}^d}t^m\partial_t^{m+1}W_t^{\mathcal{L}}(x,z)dz\right|+\left|\int_{\mathbb{R}^d}t^m\partial_t^{m+1}W_t^{\mathcal{L}}(y,z)])dz\right|\right]^{1/2} \\ &\leq C|f_{B(x_0,r_0)}|\frac{1}{t}\left(\frac{r_0}{\rho(x_0)}\right)^{\delta/2},\end{aligned}$$

with $0 < \delta < \delta_0$. By taking into account (3.7) it follows that, for each $x, y \in B(x_0, r_0)$,

$$\int_{8r_0^2}^{8\rho(x_0)^2} F_2(x,y,t)\, dt \leq C|B(x_0,r_0)|^{\alpha-1} w(B(x_0,r_0)) \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}$$

$$\times \left(\frac{\rho(x_0)}{r_0}\right)^{d(p+\alpha-1)+p\theta-\frac{\delta}{2}} \int_{8r_0^2}^{8\rho(x_0)^2} \frac{dt}{t}$$

$$\leq Cw(B(x_0,r_0))|B(x_0,r_0)|^{d(\alpha-1)} \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)},$$

provided that $\delta_0 > \delta > 2(d(p+\alpha-1)+p\theta)$.

We conclude that

$$G_{22}(f) \leq C|B(x_0,r_0)|^{\alpha} w(B(x_0,r_0)) \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.$$

We get

$$G_2(f) \leq C|B(x_0,r_0)|^{\alpha} w(B(x_0,r_0)) \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}. \tag{3.9}$$

Thus, by considering (3.8) and (3.9) the proof can be finished.

4. Proof of Theorem 1.2 for the oscillation operator $O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$

In order to prove that the operator $O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$ is bounded from $L^p(\mathbb{R}^d, w)$ into itself for every $1 < p < \infty$ and $w \in A_p^{\rho,\infty}(\mathbb{R}^d)$, we can proceed as in the proof of [45, Theorem 4] and in [2, Theorem 1.1]. We sketch the main steps of the proof.

We firstly establish the result in the unweighted case, that is, we prove that $O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$ is bounded from $L^p(\mathbb{R}^d)$ into itself for every $1 < p < \infty$. As far as we know a L^p-boundedness result for oscillation operators like the one established in [32, Corollary 4.5] has not been proved. Since $\{W_t^{\mathcal{L}}\}_{t>0}$ is not Markovian, the L^p-boundedness of $O(\{W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$ can not be deduced from [29, Theorem 3.3, (2)].

Suppose that $F : (0,\infty) \longrightarrow \mathbb{C}$ is a differentiable function. We have that

$$O(\{F(t)\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}}) = \left(\sum_{i=-\infty}^{+\infty} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} |F(\varepsilon_i) - F(\varepsilon_{i+1})|^2\right)^{1/2}$$

$$= \left(\sum_{i=-\infty}^{+\infty} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} \Big|\int_{\varepsilon_i}^{\varepsilon_{i+1}} F'(t)\, dt\Big|^2\right)^{1/2}$$

$$\leq \left(\sum_{i=-\infty}^{+\infty} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} \Big(\int_{\varepsilon_i}^{\varepsilon_{i+1}} |F'(t)|\, dt\Big)^2\right)^{1/2}$$

$$\leq C \sum_{i=-\infty}^{+\infty} \int_{t_i}^{t_{i+1}} |F'(t)|\, dt \leq C \int_0^{\infty} |F'(t)|\, dt. \tag{4.1}$$

This inequality plays an important role in our proof for the boundedness properties for $O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$.

It is easy to see that if F is a complex function defined in $(0,\infty)$ and satisfies that $O(\{F(t)\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}}) = 0$, then F is constant. The oscillation operator associated with $\{t_j\}_{j\in\mathbb{Z}}$ defines a seminorm in the space $\mathcal{F}$ of complex functions defined in $(0,\infty)$ such that $O(\{F(t)\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}}) < \infty$.

We consider the quotient space $\mathcal{F}/\sim$, where $\sim$ is the binary relation defined as follows: if $F_1, F_2 \in \mathcal{F}$ we say that $F_1 \sim F_2$ when $F_1 - F_2$ is constant. The oscillation defines a norm on $\mathcal{F}/\sim$ and $(\mathcal{F}/\sim, O(\cdot, \{t_j\}_{j\in\mathbb{Z}}))$ is a Banach space. To see the oscillation as a norm allows us to simplify our arguments. We can also understand our oscillation operators $O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$ as Banach valued singular integral operators.

In order to prove that $O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$ defines a bounded operator from $L^p(\mathbb{R}^d)$ into itself, $1 < p < \infty$, we exploit that the Schrödinger operator $\mathcal{L}$ is a nice (in some sense) perturbation of the Euclidean Laplacian.

We now explain the procedure (see [2]).

We split the region $\mathbb{R}^d \times \mathbb{R}^d$ in two parts:

$$L = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| < \rho(x)\right\}$$

and $G = (\mathbb{R}^d \times \mathbb{R}^d) \setminus L$. L and G mean local and global regions, respectively. To simplify we write $T_{\mathcal{L}} = O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$.

We decompose the operator $T_{\mathcal{L}}$ in two parts: the local part $T_{\mathcal{L},\mathrm{loc}}(f)(x) = T_{\mathcal{L}}(f\mathcal{X}_{B(x,\rho(x))})(x)$, $x \in \mathbb{R}^d$, and the global one, $T_{\mathcal{L},\mathrm{glob}} = T_{\mathcal{L}} - T_{\mathcal{L},\mathrm{loc}}$.

We define the operators $T_{-\Delta}$, $T_{-\Delta,\mathrm{loc}}$ and $T_{-\Delta,\mathrm{glob}}$ as above by replacing the Schrödinger operator by the Euclidean Laplacian.

We decompose the operator $T_{\mathcal{L}}$ as follows:

$$T_{\mathcal{L}} = (T_{\mathcal{L},\mathrm{loc}} - T_{-\Delta,\mathrm{loc}}) + T_{-\Delta,\mathrm{loc}} + T_{\mathcal{L},\mathrm{glob}}.$$

Our objective is to establish that the operators $T_{\mathcal{L},\mathrm{loc}} - T_{-\Delta,\mathrm{loc}}$, $T_{-\Delta,\mathrm{loc}}$ and $T_{\mathcal{L},\mathrm{glob}}$ are bounded from $L^p(\mathbb{R}^d)$ into itself for every $1 < p < \infty$.

We first study $T_{-\Delta,\mathrm{loc}}$. We consider the function $\phi(z) = e^{-z}$, $z \in (0,\infty)$. The Euclidean heat kernel in $\mathbb{R}^d$ is defined by

$$W_t(z) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|z|^2}{4t}} = \frac{1}{(4\pi t)^{d/2}} \phi\left(\frac{|z|^2}{4t}\right), \quad z \in \mathbb{R}^d \text{ and } t > 0.$$

By using the Faà di Bruno formula we obtain, for each $z \in \mathbb{R}^d$ and $t > 0$,

$$\begin{aligned}
\partial_t^k W_t(z) &= \sum_{j=0}^k c_j t^{-\frac{d}{2}-(k-j)} \partial_t^j \phi\left(\frac{|z|^2}{4t}\right) \\
&= \sum_{j=0}^k c_j t^{-\frac{d}{2}-(k-j)} \sum_{m_1+2m_2+\ldots+jm_j=j} d^j_{m_1,\ldots,m_j} \phi\left(\frac{|z|^2}{4t}\right) \frac{|z|^{2(m_1+\ldots+m_j)}}{t^{m_1+\ldots+m_j+j}} \\
&= \sum_{j=0}^k \sum_{m_1+2m_2+\ldots+jm_j=j} c_j d^j_{m_1,\ldots,m_j} \frac{1}{t^{\frac{d}{2}+k}} \phi\left(\frac{|z|^2}{4t}\right) \left(\frac{|z|^2}{t}\right)^{m_1+\ldots+m_j},
\end{aligned}$$

where c_j, $d^j_{m_1,\dots,m_j} \in \mathbb{R}$, $j=0,\dots,k$ and $m_1+2m_2+\ldots+jm_j=j$, $m_1,\dots,m_j\in\mathbb{N}$. Then,

$$t^k\partial_t^k W_t(z)=\frac{1}{t^{d/2}}\psi\left(\frac{|z|}{\sqrt{t}}\right),\quad z\in\mathbb{R}^d \text{ and } t>0, \tag{4.2}$$

being

$$\psi(u)=\sum_{j=0}^{k}\sum_{m_1+2m_2+\ldots+jm_j=j} c_j d^j_{m_1,\dots,m_j}\phi(u^2)u^{2(m_1+\ldots+m_j)},\quad u\in\mathbb{R}.$$

Note that ψ is in the Schwartz class $\mathcal{S}(\mathbb{R})$. According to [20, Lemma 2.4, (1)] the operator $T_{-\Delta}$ is bounded from $L^p(\mathbb{R}^d)$ into itself for every $1<p<\infty$.

According to [25, Proposition 5] we choose a sequence $\{x_j\}_{j\in\mathbb{N}}\subset\mathbb{R}^d$ such that by defining $Q_j=B(x_j,\rho(x_j))$ the following two properties hold:

(i) $\bigcup_{j\in\mathbb{N}} Q_j=\mathbb{R}^d$.

(ii) For every $m\in\mathbb{N}$, there exist $\gamma,\beta\in\mathbb{N}$ such that, for every $j\in\mathbb{N}$, the set

$$\{\ell\in\mathbb{N}: 2^mQ_\ell\cap 2^mQ_j\neq\emptyset\}$$

has at most $\gamma 2^{m\beta}$ elements.

Let $j\in\mathbb{N}$. If $x\in Q_j$ and $z\in B(x,\rho(x))$, then

$$|z-x_j|\le|z-x|+|x-x_j|\le\rho(x)+\rho(x_j)\le C_1\rho(x_j),$$

because $\rho(x)\sim\rho(x_j)$. Here C_1 does not depend on j.

We consider, for every $t>0$, the operator

$$H_t^j(f)(x)=\mathcal{X}_{Q_j}(x)\int_{B(x_j,C_1\rho(x_j))\setminus B(x,\rho(x))} t^k\partial_t^kW_t(x-y)f(y)\,dy,\quad x\in\mathbb{R}^d.$$

By using (4.1) and (4.2) we deduce that, for every $z\in\mathbb{R}^d\setminus\{0\}$,

$$\begin{aligned}O(\{t^k\partial_t^kW_t(z)\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})&\\ &\le C\int_0^\infty\left|\partial_t\left(\frac{1}{t^{d/2}}\psi\left(\frac{|z|}{\sqrt{t}}\right)\right)\right|dt\\ &\le C\int_0^\infty\frac{1}{t^{\frac{d}{2}+1}}\left(\left|\psi\left(\frac{|z|}{\sqrt{t}}\right)\right|+\frac{|z|}{\sqrt{t}}\left|\psi'\left(\frac{|z|}{\sqrt{t}}\right)\right|\right)dt\le\frac{C}{|z|^d}.\end{aligned}$$

It follows that

$$\begin{aligned}O(\{H_t^j\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})(f)(x)&\\ &\le\mathcal{X}_{Q_j}(x)\int_{B(x_j,C_1\rho(x_j))\setminus B(x,\rho(x))}O(\{t^k\partial_t^kW_t(x-y)\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})|f(y)|\,dy\\ &\le C\mathcal{X}_{Q_j}(x)\int_{B(x_j,C_1\rho(x_j))\setminus B(x,\rho(x))}\frac{1}{|x-y|^d}|f(y)|\,dy\\ &\le\frac{C}{\rho(x)^d}\mathcal{X}_{Q_j}(x)\int_{B(x_j,C_1\rho(x_j))}|f(y)|\,dy\\ &\le\frac{C}{\rho(x_j)^d}\mathcal{X}_{Q_j}(x)\int_{B(x_j,C_1\rho(x_j))}|f(y)|\,dy\le C\mathcal{X}_{Q_j}(x)\mathcal{M}_{\rm HL}(f)(x),\quad x\in\mathbb{R}^d.\end{aligned}$$

Here $\mathcal{M}_{\rm HL}$ represents the classical Hardy–Littlewood maximal operator. We have that

$$T_{-\Delta}(\mathcal{X}_{B(x_j,C_1\rho(x_j))}f)=T_{-\Delta,\rm loc}(f)(x)+T_{-\Delta}(\mathcal{X}_{B(x_j,C_1\rho(x_j))\setminus B(x,\rho(x))}f)(x),\quad x\in Q_j.$$

Then,

$$T_{-\Delta,\rm loc}(f)(x)\leq T_{-\Delta}(\mathcal{X}_{B(x_j,C_1\rho(x_j))}f)+C\mathcal{M}_{\rm HL}(f)(x),\quad x\in Q_j.$$

Let $1<p<\infty$. We can write

$$\begin{aligned}\int_{\mathbb{R}^d}|T_{-\Delta,\rm loc}(f)(x)|^p\,dx&=\sum_{j\in\mathbb{N}}\int_{Q_j}|T_{-\Delta,\rm loc}(f)(x)|^p\,dx\\ &\leq C\left(\sum_{j\in\mathbb{N}}\int_{Q_j}|T_{-\Delta}(\mathcal{X}_{B(x_j,C_1\rho(x_j))}f)(x)|^p\,dx+\int_{\mathbb{R}^d}|\mathcal{M}_{\rm HL}(f)(x)|^p\,dx\right)\\ &\leq C\left(\sum_{j\in\mathbb{N}}\int_{B(x_j,C_1\rho(x_j))}|f(x)|^p\,dx+\int_{\mathbb{R}^d}|f(x)|^p\,dx\right)\leq C\int_{\mathbb{R}^d}|f(x)|^p\,dx.\end{aligned}$$

Thus we have proved that $T_{-\Delta,\rm loc}$ is bounded from $L^p(\mathbb{R}^d)$ into itself.

By using (4.1) and Proposition 2.1 (a), proceeding as in [2, p. 506] we can deduce that

$$T_{\mathcal{L},\rm glob}(f)\leq C\mathcal{M}_{\rm HL}(f).$$

Then, $T_{\mathcal{L},\rm glob}$ is bounded from $L^p(\mathbb{R}^d)$ into itself.

The arguments in [2, pp. 507–509] by using again (4.1) and now Proposition 2.1 (d), allow us to prove that

$$|T_{\mathcal{L},\rm loc}(f)-T_{-\Delta,\rm loc}(f)|\leq C\mathcal{M}_{\rm HL}(f).$$

We conclude that $T_{\mathcal{L},\rm loc}-T_{-\Delta,\rm loc}$ is bounded from $L^p(\mathbb{R}^d)$ into itself.

By putting together all the above estimates we deduce that the oscillation operator $O(\{t^k\partial_t^kW_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})$ is bounded from $L^p(\mathbb{R}^d)$ into itself.

After proving that $O(\{t^k\partial_t^kW_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})$ is bounded from $L^p(\mathbb{R}^d)$ into itself for every $1<p<\infty$, by using the properties established in Proposition 2.1, we can proceed as in [45, pp. 605–609] to establish that $O(\{t^k\partial_t^kW_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})$ is bounded from $L^p(\mathbb{R}^d,w)$ into itself, for every $1<p<\infty$ and $w\in A_p^{\rho,\infty}(\mathbb{R}^d)$.

We are going to see that the oscillation operator $O(\{t^k\partial_t^kW_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})$ is bounded from $\mathrm{BMO}_{\mathcal{L},w}^{\alpha}(\mathbb{R}^d)$ into $\mathrm{BLO}_{\mathcal{L},w}^{\alpha}(\mathbb{R}^d)$.

By using the weighted L^p-boundedness properties of $O(\{t^k\partial_t^kW_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})$ that we have just proved and Proposition 2.2, we can prove by proceeding as in [45, pp. 610–611 and Lemma 2.1] that there exists $C>0$ for which, for every $f\in\mathrm{BMO}_{\mathcal{L},w}^{\alpha}(\mathbb{R}^d)$, and each $x_0\in\mathbb{R}^d$ and $r_0\geq\rho(x_0)$,

$$\begin{aligned}\int_{B(x_0,r_0)}&|O(\{t^k\partial_t^kW_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})(f)(x)|\,dx\\ &\leq C|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}_{\mathcal{L},w}^{\alpha}(\mathbb{R}^d)}.\end{aligned}$$

Note that the last inequality implies that, for every $f \in \mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)$, we have that

$$O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})(f)(x)<\infty \quad \text{for almost all } x\in\mathbb{R}^d.$$

To finish the proof we need to see that there exists $C>0$ such that, for every $f \in \mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)$, $x_0\in\mathbb{R}^d$ and $0<r_0<\rho(x_0)$,

$$\begin{aligned}\int_{B(x_0,r_0)}\Big(&O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})(f)(x)\\ &-\operatorname*{ess\,inf}_{y\in B(x_0,r_0)} O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})(f)(y)\Big)\,dx\\ &\leq C\||B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.\end{aligned}$$

Let $f \in \mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)$, $x_0\in\mathbb{R}^d$ and $0<r_0<\rho(x_0)$. We choose $i_0\in\mathbb{Z}$ such that $t_{i_0}<8r_0^2\leq t_{i_0+1}$.

We define the following sets:

$$\begin{aligned}D_1=\Big\{&y\in B(x_0,r_0):\\ &\sup_{t_{i_0}\leq\varepsilon_{i_0}<\varepsilon_{i_0+1}\leq t_{i_0+1}}\Big|t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0}}-t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}\Big|\\ &=\sup_{t_{i_0}\leq\varepsilon_{i_0}<\varepsilon_{i_0+1}\leq 8r_0^2}\Big|t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0}}-t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}\Big|\Big\},\end{aligned}$$

$$\begin{aligned}D_2=\Big\{&y\in B(x_0,r_0):\\ &\sup_{t_{i_0}\leq\varepsilon_{i_0}<\varepsilon_{i_0+1}\leq t_{i_0+1}}\Big|t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0}}-t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}\Big|\\ &=\sup_{8r_0^2\leq\varepsilon_{i_0}<\varepsilon_{i_0+1}\leq t_{i_0+1}}\Big|t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0}}-t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}\Big|\Big\}\end{aligned}$$

and

$$\begin{aligned}D_3=\Big\{&y\in B(x_0,r_0):\\ &\sup_{t_{i_0}\leq\varepsilon_{i_0}<\varepsilon_{i_0+1}\leq t_{i_0+1}}\Big|t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0}}-t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}\Big|\\ &=\sup_{t_{i_0}\leq\varepsilon_{i_0}<8r_0^2<\varepsilon_{i_0+1}\leq 8r_0^2}|t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0}}-t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}|\Big\}.\end{aligned}$$

We consider the following decomposition:

$$\begin{aligned}\int_{B(x_0,r_0)}\Big(&O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})(f)(x)\\ &-\operatorname*{ess\,inf}_{y\in B(x_0,r_0)} O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})(f)(y)\Big)\,dx=\sum_{i=1}^{3}H_i,\end{aligned}$$

where, for every $i = 1, 2, 3$,

$$H_i = \int_{D_i} \Big(O(\{t^k \partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(x) \\ - \operatorname*{ess\,inf}_{y\in B(x_0,r_0)} O(\{t^k \partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(y) \Big)\, dx.$$

We have, for every $y \in B(x_0, r_0)$,

$$O(\{t^k \partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(y) \\ \geq \left(\sum_{i=i_0+1}^{+\infty} \sup_{t_i\le\varepsilon_i<\varepsilon_{i+1}\le t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2 \right)^{1/2}.$$

Then,

$$H_1 \le \int_{D_1} \Big(\sum_{i=-\infty}^{i_0} \sup_{\substack{t_i\le\varepsilon_i<\varepsilon_{i+1}\le t_{i+1}\\ \varepsilon_{i+1}\le 8r_0^2}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_{i+1}}|^2 \Big)^{1/2} dx \\ + \int_{D_1} \Big[\Big(\sum_{i=i_0+1}^{+\infty} \sup_{t_i\le\varepsilon_i<\varepsilon_{i+1}\le t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2 \Big)^{1/2} \\ - \Big(\operatorname*{ess\,inf}_{y\in B(x_0,r_0)} \Big(\sum_{i=i_0+1}^{+\infty} \sup_{t_i\le\varepsilon_i<\varepsilon_{i+1}\le t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} \\ - t^k k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2 \Big)^{1/2} \Big] dx \\ \le \int_{B(x_0,r_0)} \Big(\sum_{i=-\infty}^{i_0} \sup_{\substack{t_i\le\varepsilon_i<\varepsilon_{i+1}\le t_{i+1}\\ \varepsilon_{i+1}\le 8r_0^2}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_{i+1}}|^2 \Big)^{1/2} dx \\ + |B(x_0,r_0)| \operatorname*{ess\,sup}_{z,y\in B(x_0,r_0)} \Big(\sum_{i=i_0+1}^{+\infty} \sup_{t_i\le\varepsilon_i<\varepsilon_{i+1}\le t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_i} \\ - t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_{i+1}} \Big) - \Big(t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}})|^2 \Big)^{1/2}.$$

On the other hand, we can write, for every $y \in B(x_0, r_0)$,

$$O(\{t^k \partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(y) \\ \geq \left(\sum_{i=i_0}^{+\infty} \sup_{\substack{t_i\le\varepsilon_i<\varepsilon_{i+1}\le t_{i+1}\\ \varepsilon_i\ge 8r_0^2}} |(t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2 \right)^{1/2}.$$

It follows that

$$H_2 \leq \int_{B(x_0,r_0)} \Big(\sum_{i=-\infty}^{i_0-1} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2 \Big)^{1/2} dy$$
$$+ |B(x_0,r_0)|$$
$$\times \operatorname*{ess\,sup}_{z,y\in B(x_0,r_0)} \Big(\sum_{i=i_0}^{+\infty} \sup_{\substack{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1} \\ \varepsilon_i \geq 8r_0^2}} |(t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_{i+1}})$$
$$- (t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}})|^2 \Big)^{1/2}.$$

Finally, in order to estimate H_3, we observe that, when $y \in D_3$,

$$O(\{t^k \partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})(f)(y)$$
$$\leq \Big(\sum_{i=-\infty}^{i_0-1} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2$$
$$+ \Big(\sup_{t_{i_0} \leq \varepsilon_{i_0} \leq 8r_0^2} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=t_{i_0}} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=8r_0^2}|$$
$$+ \sup_{8r_0^2 \leq \varepsilon_{i_0+1} \leq t_{i_0+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=8r_0^2} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=t_{i_0+1}}| \Big)^2$$
$$+ \sum_{i=i_0+1}^{+\infty} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2 \Big)^{1/2}$$
$$\leq \Big(\sum_{i=-\infty}^{i_0-1} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2$$
$$+ \sup_{t_{i_0} \leq \varepsilon_{i_0} \leq 8r_0^2} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=8r_0^2}|^2 \Big)^{1/2}$$
$$+ \Big(\sup_{8r_0^2 \leq \varepsilon_{i_0+1} \leq t_{i_0+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=8r_0^2} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}|^2$$
$$+ \sum_{i=i_0+1}^{+\infty} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2 \Big)^{1/2}$$
$$\leq \Big(\sum_{i=-\infty}^{i_0-1} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2$$
$$+ \sup_{t_{i_0} \leq \varepsilon_{i_0} < \varepsilon_{i_0+1} \leq 8r_0^2} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0}} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}|^2 \Big)^{1/2}$$

$$+\Big(\sup_{8r_0^2\leq\varepsilon_{i_0+1}\leq t_{i_0+1}}|t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=8r_0^2}-t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}|^2$$
$$+\sum_{i=i_0+1}^{+\infty}\sup_{t_i\leq\varepsilon_i<\varepsilon_{i+1}\leq t_{i+1}}|t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i}-t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2\Big)^{1/2}.$$

Thus, we deduce that, for every $y\in D_3$,

$$O(\{t^k\partial_t^kW_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})(f)(y)$$
$$\leq\Big(\sum_{i=-\infty}^{i_0}\sup_{\substack{t_i\leq\varepsilon_i<\varepsilon_{i+1}\leq t_{i+1}\\ \varepsilon_{i+1}\leq 8r_0^2}}|t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i}-t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2\Big)^{1/2}$$
$$+\Big(\sup_{8r_0^2\leq\varepsilon_{i_0+1}\leq t_{i_0+1}}|t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=8r_0^2}-t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2$$
$$+\sum_{i=i_0+1}^{+\infty}\sup_{t_i\leq\varepsilon_i<\varepsilon_{i+1}\leq t_{i+1}}|t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i}-t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2\Big)^{1/2}.$$

On the other hand we have that, for each $y\in B(x_0,r_0)$,

$$O(\{t^k\partial_t^kW_t^{\mathcal{L}}\}_{t>0},\{t_j\}_{j\in\mathbb{Z}})(f)(y)$$
$$\geq\Big(\sum_{i=i_0+1}^{\infty}\sup_{t_i\leq\varepsilon_i<\varepsilon_{i+1}\leq t_{i+1}}|(t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i}-t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2$$
$$+\sup_{t_{i_0}\leq\varepsilon_{i_0}\leq 8r_0^2<\varepsilon_{i_0+1}\leq t_{i_0+1}}|t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0}}-t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}|^2\Big)^{1/2}$$
$$\geq\Big(\sum_{i=i_0+1}^{\infty}\sup_{t_i\leq\varepsilon_i<\varepsilon_{i+1}\leq t_{i+1}}|(t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i}-t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}}|^2$$
$$+\sup_{8r_0^2<\varepsilon_{i_0+1}\leq t_{i_0+1}}|t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=8r_0^2}-t^k\partial_t^kW_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}}|^2\Big)^{1/2}.$$

It follows that

$$
\begin{aligned}
H_3 \leq & \int_{D_3} \left(\sum_{i=-\infty}^{i_0} \sup_{\substack{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1} \\ \varepsilon_{i+1} \leq 8r_0^2}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_{i+1}}|^2 \right)^{1/2} dx \\
& + |B(x_0, r_0)| \\
& \times \operatorname*{ess\,sup}_{z,y \in B(x_0,r_0)} \left(\sum_{i=i_0+1}^{+\infty} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} | \left(t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_{i+1}} \right) \right. \\
& \quad - \left(t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}} \right) |^2 \\
& \quad + \sup_{8r_0^2 < \varepsilon_{i_0+1} \leq t_{i_0+1}} | \Big(t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=8r_0^2} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_{i_0+1}} \Big) \\
& \quad \left. - \Big(t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=8r_0^2} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}} \Big) |^2 \right)^{1/2} \\
\leq & \int_{D_3} \left(\sum_{i=-\infty}^{i_0} \sup_{\substack{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1} \\ \varepsilon_{i+1} \leq 8r_0^2}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_{i+1}}|^2 \right)^{1/2} dx \\
& + |B(x_0, r_0)| \\
& \times \operatorname*{ess\,sup}_{z,y \in B(x_0,r_0)} \left(\sum_{i=i_0+1}^{+\infty} \sup_{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1}} |(t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_{i+1}}) \right. \\
& \quad - (t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}})|^2 \\
& \quad + \sup_{8r_0^2 \leq \varepsilon_{i_0} < \varepsilon_{i_0+1} \leq t_{i_0+1}} |(t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_{i_0}} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_{i_0+1}}) \\
& \quad \left. - \Big(t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=8r_0^2} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i_0+1}} \Big) |^2 \right)^{1/2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
H_3 \leq & \int_{B(x_0,r_0)} \left(\sum_{i=-\infty}^{i_0} \sup_{\substack{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1} \\ \varepsilon_{i+1} \leq 8r_0^2}} |t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_{i+1}}|^2 \right)^{1/2} dx \\
& + |B(x_0, r_0)| \\
& \quad \times \operatorname*{ess\,sup}_{,z,y \in B(x_0,r_0)} \left(\sum_{i=i_0}^{+\infty} \sup_{\substack{t_i \leq \varepsilon_i < \varepsilon_{i+1} \leq t_{i+1} \\ \varepsilon_i \geq 8r_0^2}} | \left(t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(z)_{|t=\varepsilon_{i+1}} \right) \right. \\
& \qquad \qquad \left. - \left(t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k \partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}} \right) |^2 \right)^{1/2}.
\end{aligned}
$$

In order to get our objective it is sufficient to prove that

$$\int_{B(x_0,r_0)} \Big(\sum_{i=-\infty}^{i_0} \sup_{\substack{t_i\leq\varepsilon_i<\varepsilon_{i+1}\leq t_{i+1}\\ \varepsilon_{i+1}\leq 8r_0^2}} |t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_i} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_{i+1}}|^2\Big)^{1/2} dx$$
$$\leq C|B(x_0,r_0)|^\alpha w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)}, \tag{4.3}$$

and

$$\operatorname*{ess\,sup}_{x,y\in B(x_0,r_0)} \Big(\sum_{i=i_0}^{+\infty} \sup_{\substack{t_i\leq\varepsilon_i<\varepsilon_{i+1}\leq t_{i+1}\\ \varepsilon_{i}\geq 8r_0^2}} |(t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_i} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=\varepsilon_{i+1}})$$
$$- (t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_i} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)_{|t=\varepsilon_{i+1}})|^2\Big)^{1/2}$$
$$\leq C|B(x_0,r_0)|^{\alpha-1} w(B(x_0,r_0)\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)}. \tag{4.4}$$

By using (4.1) we can get (4.3) and (4.4) by proceeding as in the proof of (3.8) and (3.9), respectively.

5. Proof of Theorem 1.2 for the operator $SV(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})$

In order to prove Theorem 1.2 for the short variation operator $SV(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})$ we can proceed in the same way to the previous section for the oscillation operator $O(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0}, \{t_j\}_{j\in\mathbb{Z}})$.

Note firstly that if F is a differentiable function in $(0,\infty)$ we have that

$$SV(\{F(t)\}_{t>0}) \leq C\int_0^\infty |F'(t)|\,dt. \tag{5.1}$$

By taking into account (4.2) and according to [20, Lemma 2.4, (3)] it follows that the operator $SV(\{t^k\partial_t^k W_t\}_{t>0})$ is bounded from $L^p(\mathbb{R}^d)$ into itself, for every $1<p<\infty$.

We now define the local and global operators as in section 4. We have that

$$SV(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f) \leq SV_{\mathrm{loc}}(\{t^k\partial_t^k (W_t^{\mathcal{L}} - W_t)\}_{t>0})(f) + SV_{\mathrm{loc}}(\{t^k\partial_t^k W_t\}_{t>0})(f)$$
$$+ SV_{\mathrm{glob}}(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f).$$

Then, by proceeding as in the study of the oscillation operator in the previous section we can see that $SV(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})$ is bounded from $L^p(\mathbb{R}^d)$ into itself, for every $1<p<\infty$. By using (5.1) the arguments in [45, pp. 605–609] allow us to see that the operator $SV\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})$ is bounded from $L^p(\mathbb{R}^d, w)$ into itself, for every $1<p<\infty$ and $w\in A_p^{\rho,\infty}(\mathbb{R}^d)$.

Let now $x_0\in\mathbb{R}^d$ and $r_0>0$ such that $r_0<\rho(x_0)$. We choose $k_0\in\mathbb{N}$ such that $2^{-k_0} < 8r_0^2 \leq 2^{-k_0+1}$. We have that

$$V_{k_0}(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(x)$$
$$= \sup_{\substack{2^{-k_0}<t_n<\ldots<t_1\leq 2^{-k_0+1}\\ n\in\mathbb{N}}} \left(\sum_{j=1}^{n-1} |t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_j} - t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=t_{j+1}}|^2\right)^{1/2}$$

$$
\begin{aligned}
&\leq \sup_{\substack{2^{-k_0}<s_\ell<\ldots<s_1\leq 8r_0^2\\ \ell\in\mathbb{N}}} \left(\sum_{j=1}^{\ell-1}|t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=s_j}-t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=s_{j+1}}|^2\right)^{1/2}\\
&\quad+\sup_{\substack{8r_0^2\leq s_\ell<\ldots<s_1\leq 2^{-k_0+1}\\ \ell\in\mathbb{N}}} \left(\sum_{j=1}^{\ell-1}|t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=s_j}-t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)_{|t=s_{j+1}}|^2\right)^{1/2}\\
&=: V_{k_0,-}(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(x)+V_{k_0,+}(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(x),\quad x\in\mathbb{R}^d,
\end{aligned}
$$

and

$$
V_{k_0}(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(x)\geq V_{k_0,+}(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(x),\quad x\in\mathbb{R}^d.
$$

It follows that

$$
\begin{aligned}
&\int_{B(x_0,r_0)}(SV(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(x)-\operatorname*{ess\,inf}_{y\in B(x_0,r_0)} SV(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(y))\,dx\\
&\leq \int_{B(x_0,r_0)}\left(\sum_{j=k_0+1}^{\infty}(V_j(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(x))^2+(V_{k_0,-}(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(x))^2\right)^{1/2}dx\\
&\quad+|B(x_0,r_0)|\operatorname*{ess\,sup}_{z,y\in B(x_0,r_0)}\Bigg|\Big((V_{k_0,+}(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(z))^2\\
&\qquad+\sum_{j=-\infty}^{k_0-1}(V_j(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(z))^2\Big)^{1/2}\\
&\qquad-\Big((V_{k_0,+}(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(y))^2+\sum_{j=-\infty}^{k_0-1}(V_j(\{t^k\partial_t^k W_t^{\mathcal{L}}\}_{t>0})(f)(y))^2\Big)^{1/2}\Bigg|.
\end{aligned}
$$

We have all the ingredients to finish the proof by proceeding as in section 3.

6. Proof of Theorem 1.1

We firstly establish that the maximal operator $W_*^{\mathcal{L},k}$ is bounded from $L^p(\mathbb{R}^d,w)$ into itself. In order to do this, it is sufficient to proceed as in the proof of [14, Theorem 2] by using Proposition 2.1, (a).

Let $f\in \mathrm{BMO}_{\mathcal{L},w}^{\alpha}(\mathbb{R}^d)$ and $x_0\in\mathbb{R}^d$. Taking $r_0=\rho(x_0)$, we decompose f as follows:

$$
\begin{aligned}
f&=f\mathcal{X}_{B(x_0,2r_0)}+f\mathcal{X}_{B(x_0,2r_0)^c}\\
&=: f_1+f_2.
\end{aligned}
$$

Since $w\in A_p^{\rho,\theta}(\mathbb{R}^d)$, we have that $w^{-1/(p-1)}\in A_{p'}^{\rho,\theta}(\mathbb{R}^d)$ (Proposition 2.2 (a)). Hölder's inequality and Propositions 2.2 (c) and 2.3 lead to

$$\begin{aligned}\int_{B(x_0,r_0)} &|W_*^{\mathcal{L},k}(f_1)(x)|\,dx \\ &\leq w(B(x_0,r_0))^{1/p}\left(\int_{B(x_0,r_0)}|W_*^{\mathcal{L},k}(f_1)(x)|^{p'}w^{-\frac{1}{p-1}}(x)\,dx\right)^{1/p'} \\ &\leq Cw(B(x_0,r_0))^{1/p}\left(\int_{B(x_0,2r_0)}|f(x)|^{p'}w^{-\frac{1}{p-1}}(x)\,dx\right)^{1/p'} \\ &\leq C|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.\end{aligned}$$

On the other hand, by using Proposition 2.1 for every $N\in\mathbb{N}$ we can find $C=C(N)>0$ such that, for each $x\in B(x_0,r_0)$,

$$\begin{aligned}|W_*^{\mathcal{L},k}(f_2)(x)| &\leq C\sup_{t>0}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{-N}\frac{1}{t^{d/2}}\int_{\mathbb{R}^d\setminus B(x_0,2r_0)}e^{-c\frac{|x-y|^2}{t}}|f(y)|\,dy \\ &\leq C\rho(x_0)^N\int_{\mathbb{R}^d\setminus B(x_0,2r_0)}\frac{|f(y)|}{|x_0-y|^{N+d}}|f(y)|\,dy \\ &\leq C\rho(x_0)^N\sum_{j=1}^{\infty}\frac{1}{(2^j\rho(x_0))^{N+d}}\int_{B(x_0,2^{j+1}r_0)}|f(y)|\,dy \\ &\leq \frac{C}{r_0^d}\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}\sum_{j=1}^{\infty}\frac{1}{2^{j(N+d)}}|B(x_0,2^{j+1}r_0)|^{\alpha}w(B(x_0,2^{j+1}r_0)) \\ &\leq C\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}|B(x_0,r_0)|^{\alpha-1}w(B(x_0,r_0))\sum_{j=1}^{\infty}2^{j(\alpha d+(\theta+d)p-d-N)}.\end{aligned}$$

In the last inequality we have used Proposition 2.2 (b). By taking $N\in\mathbb{N}$, $N>d(p+\alpha-1)+p\theta$ we obtain

$$|W_*^{\mathcal{L},k}(f_2)(x)|\leq C|B(x_0,r_0)|^{\alpha-1}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)},\quad x\in B(x_0,r_0).$$

Then,

$$\int_{B(x_0,r_0)}|W_*^{\mathcal{L},k}(f_2)(x)|\,dx\leq C|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)},$$

and we conclude that

$$\int_{B(x_0,r_0)}|W_*^{\mathcal{L},k}(f)(x)|\,dx\leq C|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}. \tag{6.1}$$

From (6.1) we deduce that $W_*^{\mathcal{L},k}(f)(x)<\infty$ for almost all $x\in\mathbb{R}^d$.

Let now $x_0\in\mathbb{R}^d$ and $0<r_0<\rho(x_0)$. We are going to see that

$$\begin{aligned}\int_{B(x_0,r_0)}&(W_*^{\mathcal{L},k}(f)(x)-\operatorname*{ess\,inf}_{y\in B(x_0,r_0)}W_*^{\mathcal{L},k}(f)(y))\,dx \\ &\leq C|B(x_0,r_0)|^{\alpha}w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.\end{aligned}$$

In order to do this we adapt the ideas developed in section 3. We have that

$$\begin{aligned}
&\int_{B(x_0,r_0)} (W_*^{\mathcal{L},k}(f)(x) - \operatorname*{ess\,inf}_{y\in B(x_0,r_0)} W_*^{\mathcal{L},k}(f)(y))\,dx \\
&\quad \leq \int_{B(x_0,r_0)} \sup_{0<t<8r_0^2} |t^k\partial_t^k W_t^{\mathcal{L}}(f)(x)|\,dx \\
&\qquad + |B(x_0,r_0)| \operatorname*{ess\,sup}_{z,y\in B(x_0,r_0)} \sup_{t\geq 8r_0^2} |t^k\partial_t^k W_t^{\mathcal{L}}(f)(z) - t^k\partial_t^k W_t^{\mathcal{L}}(f)(y)| \\
&\quad =: M_1(f) + M_2(f).
\end{aligned}$$

We decompose f as follows:

$$\begin{aligned}
f &= (f - f_{B(x_0,r_0)})\mathcal{X}_{B(x_0,2r_0)} + (f - f_{B(x_0,r_0)})\mathcal{X}_{B(x_0,2r_0)^c} + f_{B(x_0,r_0)} \\
&=: f_1 + f_2 + f_3.
\end{aligned}$$

Since $W_*^{\mathcal{L},k}$ is bounded from $L^{p'}(\mathbb{R}^d, w^{-1/(p-1)})$ into itself we get

$$M_1(f_1) \leq C|B(x_0,r_0)|^\alpha w(B(x_0,r_0)\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)}.$$

According to Proposition 2.1, (a), we obtain

$$\begin{aligned}
M_1(f_2) &\leq C\int_{B(x_0,r_0)}\int_{\mathbb{R}^d\setminus B(x_0,2r_0)} |f(y) - f_{B(x_0,r_0)}| \sup_{0<t<8r_0^2} \frac{1}{t^{d/2}} e^{-c\frac{|x-y|^2}{t}}\,dydx \\
&\leq C\int_{\mathbb{R}^d\setminus B(x_0,2r_0)} |f(y) - f_{B(x_0,r_0)}| \frac{e^{-c\frac{|x_0-y|^2}{r_0^2}}}{|x_0-y|^d}\,dydx \\
&\leq C|B(x_0,r_0)|^\alpha w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)}.
\end{aligned}$$

Suppose now $k\in\mathbb{N}$, $k\geq 1$. Since $\partial_t^k W_t(1) = 0$, it follows that

$$\begin{aligned}
M_1(f_3) &\leq |f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \sup_{0<t<8r_0^2} \Big|\int_{\mathbb{R}^d} t^k\partial_t^k[W_t^{\mathcal{L}}(x,y) - W_t(x-y)]dy\Big|\,dx \\
&\leq |f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \sup_{0<t<8r_0^2} \int_{|x-y|<\rho(x_0)} |t^k\partial_t^k[W_t^{\mathcal{L}}(x,y) - W_t(x-y)]|\,dydx \\
&\quad + |f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \sup_{0<t<8r_0^2} \int_{|x-y|\geq\rho(x_0)} |t^k\partial_t^k[W_t^{\mathcal{L}}(x,y) - W_t(x-y)]|\,dydx \\
&=: M_{11}(f_3) + M_{12}(f_3).
\end{aligned}$$

According to Proposition 2.1, (d), since $2 - \frac{d}{q} > d(p+\alpha-1) + p\theta$, we obtain

$$\begin{aligned}
M_{11}(f_3) &\leq C|f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \sup_{0<t<8r_0^2} \int_{|x-y|<\rho(x_0)} \Big(\frac{\sqrt{t}}{\rho(x_0)}\Big)^{2-\frac{d}{q}} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}}\,dydx \\
&\leq C|f_{B(x_0,r_0)}| \int_{B(x_0,r_0)} \int_{|x-y|<\rho(x_0)} \rho(x_0)^{\frac{d}{q}-2} \frac{e^{-c\frac{|x-y|^2}{r_0^2}}}{|x-y|^{d+\frac{d}{q}-2}}\,dydx \\
&\leq C|B(x_0,r_0)|^\alpha w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)}.
\end{aligned}$$

By using again Proposition 2.1, (a), and (3.5), for every $\beta > 0$, we get

$$\int_{B(x_0,r_0)} \sup_{0<t<8r_0^2} \int_{|x-y|\geq\rho(x_0)} |t^k\partial_t^k[W_t^{\mathcal{L}}(x,y) - W_t(x-y)]|\, dydx \leq C|B(x_0,r_0)|\Big(\frac{r_0}{\rho(x_0)}\Big)^{\beta}.$$

By taking $\beta = d(\alpha+p-1)+p\theta$ it follows that

$$M_{12}(f_3) \leq C|B(x_0,r_0)|^{\alpha} w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.$$

We conclude that

$$M_1(f_3) \leq C|B(x_0,r_0)|^{\alpha} w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.$$

We can write

$$\begin{aligned} M_2(f) &\leq |B(x_0,r_0)| \operatorname*{ess\,sup}_{x,y\in B(x_0,r_0)} \sup_{t>8\rho(x_0)^2} \Big|\int_{\mathbb{R}^d} [t^k\partial_t^k W_t^{\mathcal{L}}(x,z) - t^k\partial_t^k W_t^{\mathcal{L}}(y,z)]f(z)dz\Big| \\ &\quad + |B(x_0,r_0)| \operatorname*{ess\,sup}_{x,y\in B(x_0,r_0)} \sup_{8r_0^2\leq t<8\rho(x_0)^2} \Big|\int_{\mathbb{R}^d} [t^k\partial_t^k W_t^{\mathcal{L}}(x,z) - t^k\partial_t^k W_t^{\mathcal{L}}(y,z)]f(z)dz\Big| \\ &=: M_{21}(f) + M_{22}(f). \end{aligned}$$

By using Proposition 2.1, (b), for every $0<\delta<\delta_0$, there exists $C>0$ such that

$$\Big|\int_{\mathbb{R}^d} [t^k\partial_t^k W_t^{\mathcal{L}}(x,z) - t^k\partial_t^k W_t^{\mathcal{L}}(y,z)]f(z)dz\Big| \leq C\left(\frac{|x-y|}{\sqrt{t}}\right)^{\delta} \|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)} \frac{w(B(x_0,r_0))}{r_0^{p(\theta+d)}} t^{\frac{d}{2}(p+\alpha-1)+\frac{p\theta}{2}},$$

for each $t > 8\rho(x_0)^2$ and $x, y \in B(x_0,r_0)$. Then,

$$M_{21}(f) \leq C|B(x_0,r_0)|^{\alpha} w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)},$$

provided that $\delta > d(p+\alpha-1)+p\theta$.

On the other hand, we have that

$$\begin{aligned} &\Big|\int_{\mathbb{R}^d} [t^k\partial_t^k W_t^{\mathcal{L}}(x,z) - t^k\partial_t^k W_t^{\mathcal{L}}(y,z)]f(z)dz\Big| \\ &\quad\leq \Big|\int_{\mathbb{R}^d} [t^k\partial_t^k W_t^{\mathcal{L}}(x,z) - t^k\partial_t^k W_t^{\mathcal{L}}(y,z)](f(z) - f_{B(x_0,r_0)})dz\Big| \\ &\qquad + \Big|\int_{\mathbb{R}^d} [t^k\partial_t^k W_t^{\mathcal{L}}(x,z) - t^k\partial_t^k W_t^{\mathcal{L}}(y,z)]dz\Big| |f_{B(x_0,r_0)}| \\ &\quad=: H_1(x,y,t) + H_2(x,y,t), \quad x, y \in B(x_0,r_0) \text{ and } t \in (8r_0^2, 8\rho(x_0)^2). \end{aligned}$$

We get

$$\sup_{r_0^2<t\leq 8\rho(x_0)^2} (H_1(x,y,t) + H_2(x,y,t)) \leq Cw(B(x_0,r_0))r_0^{d(\alpha-1)}\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.$$

We conclude that

$$M_{22}(f) \leq C|B(x_0,r_0)|^{\alpha} w(B(x_0,r_0))\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L},w}(\mathbb{R}^d)}.$$

Thus,

$$M_2(f) \leq C|B(x_0, r_0)|^\alpha w(B(x_0, r_0))\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)},$$

and the proof is finished when $k \in \mathbb{N}$, $k \geq 1$.

In order to establish the result for $k = 0$, that is, to see that the maximal operator $W^{\mathcal{L}}_*$ is bounded from $\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ into $\mathrm{BLO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$, we can proceed as in the proof of [51, Theorem 3.1]. We remark that the arguments in the proof of [51, Theorem 3.1] can be adapted to establish that the maximal operator $W^{\mathcal{L},k}_*$, $k \in \mathbb{N}$, $k \geq 1$, is bounded from $\mathrm{BMO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ into $\mathrm{BLO}^\alpha_{\mathcal{L},w}(\mathbb{R}^d)$ but we have preferred to show that the procedure in section 3 also works for $W^{\mathcal{L},k}_*$, $k \in \mathbb{N}$, $k \geq 1$.

References

[1] D. Beltran, R. Oberlin, L. Roncal, A. Seeger, and B. Stovall, Variation bounds for spherical averages, *Math. Ann.* **382** no. 1-2 (2022), 459–512. DOI MR Zbl

[2] J. J. Betancor, J. C. Fariña, E. Harboure, and L. Rodríguez-Mesa, L^p-boundedness properties of variation operators in the Schrödinger setting, *Rev. Mat. Complut.* **26** no. 2 (2013), 485–534. DOI MR Zbl

[3] J. J. Betancor, J. C. Fariña, E. Harboure, and L. Rodríguez-Mesa, Variation operators for semigroups and Riesz transforms on *BMO* in the Schrödinger setting, *Potential Anal.* **38** no. 3 (2013), 711–739. DOI MR Zbl

[4] B. Bongioanni, A. Cabral, and E. Harboure, Extrapolation for classes of weights related to a family of operators and applications, *Potential Anal.* **38** no. 4 (2013), 1207–1232. DOI MR Zbl

[5] B. Bongioanni, A. Cabral, and E. Harboure, Lerner's inequality associated to a critical radius function and applications, *J. Math. Anal. Appl.* **407** no. 1 (2013), 35–55. DOI MR Zbl

[6] B. Bongioanni, A. Cabral, and E. Harboure, Schrödinger type singular integrals: weighted estimates for $p = 1$, *Math. Nachr.* **289** no. 11-12 (2016), 1341–1369. DOI MR Zbl

[7] B. Bongioanni, E. Harboure, and P. Quijano, Weighted inequalities for Schrödinger type singular integrals, *J. Fourier Anal. Appl.* **25** no. 3 (2019), 595–632. DOI MR Zbl

[8] B. Bongioanni, E. Harboure, and P. Quijano, Two weighted inequalities for operators associated to a critical radius function, *Illinois J. Math.* **64** no. 2 (2020), 227–259. DOI MR Zbl

[9] B. Bongioanni, E. Harboure, and P. Quijano, Weighted inequalities of Fefferman-Stein type for Riesz-Schrödinger transforms, *Math. Inequal. Appl.* **23** no. 3 (2020), 775–803. DOI MR Zbl

[10] B. Bongioanni, E. Harboure, and P. Quijano, Fractional powers of the Schrödinger operator on weigthed Lipschitz spaces, *Rev. Mat. Complut.* **35** no. 2 (2022), 515–543. DOI MR Zbl

[11] B. Bongioanni, E. Harboure, and P. Quijano, Behaviour of Schrödinger Riesz transforms over smoothness spaces, *J. Math. Anal. Appl.* **517** no. 2 (2023), Paper No. 126613, 31 pp. DOI MR Zbl

[12] B. Bongioanni, E. Harboure, and O. Salinas, Weighted inequalities for negative powers of Schrödinger operators, *J. Math. Anal. Appl.* **348** no. 1 (2008), 12–27. DOI MR Zbl

[13] B. Bongioanni, E. Harboure, and O. Salinas, Riesz transforms related to Schrödinger operators acting on BMO type spaces, *J. Math. Anal. Appl.* **357** no. 1 (2009), 115–131. DOI MR Zbl

[14] B. Bongioanni, E. Harboure, and O. Salinas, Classes of weights related to Schrödinger operators, *J. Math. Anal. Appl.* **373** no. 2 (2011), 563–579. DOI MR Zbl

[15] B. Bongioanni, E. Harboure, and O. Salinas, Commutators of Riesz transforms related to Schrödinger operators, *J. Fourier Anal. Appl.* **17** no. 1 (2011), 115–134. DOI MR Zbl

[16] B. Bongioanni, E. Harboure, and O. Salinas, Weighted inequalities for commutators of Schrödinger-Riesz transforms, *J. Math. Anal. Appl.* **392** no. 1 (2012), 6–22. DOI MR Zbl

[17] J. Bourgain, Pointwise ergodic theorems for arithmetic sets, *Inst. Hautes Études Sci. Publ. Math.* no. 69 (1989), 5–45, With an appendix by the author, H. Furstenberg, Y. Katznelson and D. S. Ornstein. MR Zbl Available at http://www.numdam.org/item?id=PMIHES_1989__69__5_0.

[18] M. Bramanti, L. Brandolini, E. Harboure, and B. Viviani, Global $W^{2,p}$ estimates for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition, *Ann. Mat. Pura Appl. (4)* **191** no. 2 (2012), 339–362. DOI MR Zbl

[19] T. A. Bui, Boundedness of variation operators and oscillation operators for certain semigroups, *Nonlinear Anal.* **106** (2014), 124–137. DOI MR Zbl

[20] J. T. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, Oscillation and variation for the Hilbert transform, *Duke Math. J.* **105** no. 1 (2000), 59–83. DOI MR Zbl

[21] J. T. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, Oscillation and variation for singular integrals in higher dimensions, *Trans. Amer. Math. Soc.* **355** no. 5 (2003), 2115–2137. DOI MR Zbl

[22] Y. Do, C. Muscalu, and C. Thiele, Variational estimates for paraproducts, *Rev. Mat. Iberoam.* **28** no. 3 (2012), 857–878. DOI MR Zbl

[23] J. Duoandikoetxea, *Fourier Analysis*, Graduate Studies in Mathematics 29, American Mathematical Society, Providence, RI, 2001, Translated and revised from the 1995 Spanish original by David Cruz-Uribe. DOI MR

[24] X. T. Duong, L. Yan, and C. Zhang, On characterization of Poisson integrals of Schrödinger operators with BMO traces, *J. Funct. Anal.* **266** no. 4 (2014), 2053–2085. DOI MR Zbl

[25] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, *BMO* spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality, *Math. Z.* **249** no. 2 (2005), 329–356. DOI MR Zbl

[26] J. Dziubański and J. Zienkiewicz, Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality, *Rev. Mat. Iberoamericana* **15** no. 2 (1999), 279–296. DOI MR Zbl

[27] E. Harboure, O. Salinas, and B. Viviani, Boundedness of operators related to a degenerate Schrödinger semigroup, *Potential Anal.* **57** no. 3 (2022), 401–431. DOI MR Zbl

[28] J. Huang, P. Li, and Y. Liu, Regularity properties of the heat kernel and area integral characterization of Hardy space $H^1_{\mathcal{L}}$ related to degenerate Schrödinger operators, *J. Math. Anal. Appl.* **466** no. 1 (2018), 447–470. DOI MR Zbl

[29] R. L. Jones and K. Reinhold, Oscillation and variation inequalities for convolution powers, *Ergodic Theory Dynam. Systems* **21** no. 6 (2001), 1809–1829. DOI MR Zbl

[30] R. L. Jones, A. Seeger, and J. Wright, Strong variational and jump inequalities in harmonic analysis, *Trans. Amer. Math. Soc.* **360** no. 12 (2008), 6711–6742. DOI MR Zbl

[31] L. D. Ky, On weak*-convergence in $H^1_L(\mathbb{R}^d)$, *Potential Anal.* **39** no. 4 (2013), 355–368. DOI MR Zbl

[32] C. Le Merdy and Q. Xu, Strong q-variation inequalities for analytic semigroups, *Ann. Inst. Fourier (Grenoble)* **62** no. 6 (2012), 2069–2097. DOI MR Zbl

[33] D. Lépingle, La variation d'ordre p des semi-martingales, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **36** no. 4 (1976), 295–316. DOI MR Zbl

[34] C.-C. Lin and H. Liu, $\mathrm{BMO}_L(\mathbb{H}^n)$ spaces and Carleson measures for Schrödinger operators, *Adv. Math.* **228** no. 3 (2011), 1631–1688. DOI MR Zbl

[35] T. Ma, P. R. Stinga, J. L. Torrea, and C. Zhang, Regularity estimates in Hölder spaces for Schrödinger operators via a $T1$ theorem, *Ann. Mat. Pura Appl. (4)* **193** no. 2 (2014), 561–589. DOI MR Zbl

[36] T. Ma, J. L. Torrea, and Q. Xu, Weighted variation inequalities for differential operators and singular integrals, *J. Funct. Anal.* **268** no. 2 (2015), 376–416. DOI MR Zbl

[37] A. Mas and X. Tolsa, Variation for the Riesz transform and uniform rectifiability, *J. Eur. Math. Soc. (JEMS)* **16** no. 11 (2014), 2267–2321. DOI MR Zbl

[38] M. Mirek, E. M. Stein, and P. Zorin-Kranich, Jump inequalities for translation-invariant operators of Radon type on $\mathbb{Z}^d$, *Adv. Math.* **365** (2020), 107065, 57 pp. DOI MR Zbl

[39] M. Mirek, B. Trojan, and P. Zorin-Kranich, Variational estimates for averages and truncated singular integrals along the prime numbers, *Trans. Amer. Math. Soc.* **369** no. 8 (2017), 5403–5423. DOI MR Zbl

[40] R. Oberlin, A. Seeger, T. Tao, C. Thiele, and J. Wright, A variation norm Carleson theorem, *J. Eur. Math. Soc. (JEMS)* **14** no. 2 (2012), 421–464. DOI MR Zbl

[41] J. Qian, The p-variation of partial sum processes and the empirical process, *Ann. Probab.* **26** no. 3 (1998), 1370–1383. DOI MR Zbl

[42] Z. W. Shen, L^p estimates for Schrödinger operators with certain potentials, *Ann. Inst. Fourier (Grenoble)* **45** no. 2 (1995), 513–546. MR Zbl Available at http://www.numdam.org/item?id=AIF_1995__45_2_513_0.

[43] Z. Shen, On fundamental solutions of generalized Schrödinger operators, *J. Funct. Anal.* **167** no. 2 (1999), 521–564. DOI MR Zbl

[44] L. Tang, Weighted norm inequalities for Schrödinger type operators, *Forum Math.* **27** no. 4 (2015), 2491–2532. DOI MR Zbl

[45] L. Tang and Q. Zhang, Variation operators for semigroups and Riesz transforms acting on weighted L^p and BMO spaces in the Schrödinger setting, *Rev. Mat. Complut.* **29** no. 3 (2016), 559–621. DOI MR Zbl

[46] N. T. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Tracts in Mathematics 100, Cambridge University Press, Cambridge, 1992. MR

[47] Z. Wang, P. Li, and C. Zhang, Boundedness of operators generated by fractional semigroups associated with Schrödinger operators on Campanato type spaces via $T1$ theorem, *Banach J. Math. Anal.* **15** no. 4 (2021), Paper No. 64, 37 pp. DOI MR Zbl

[48] L. Wu and L. Yan, Heat kernels, upper bounds and Hardy spaces associated to the generalized Schrödinger operators, *J. Funct. Anal.* **270** no. 10 (2016), 3709–3749. DOI MR Zbl

[49] D. Yang, D. Yang, and Y. Zhou, Endpoint properties of localized Riesz transforms and fractional integrals associated to Schrödinger operators, *Potential Anal.* **30** no. 3 (2009), 271–300. DOI MR Zbl

[50] D. Yang, D. Yang, and Y. Zhou, Localized BMO and BLO spaces on RD-spaces and applications to Schrödinger operators, *Commun. Pure Appl. Anal.* **9** no. 3 (2010), 779–812. DOI MR Zbl

[51] D. Yang, D. Yang, and Y. Zhou, Localized Morrey-Campanato spaces on metric measure spaces and applications to Schrödinger operators, *Nagoya Math. J.* **198** (2010), 77–119. DOI MR Zbl

[52] K. Yosida, *Functional Analysis*, fifth ed., Grundlehren der Mathematischen Wissenschaften 123, Springer-Verlag, Berlin-New York, 1978. MR Zbl

[53] Q. Zhang and L. Tang, Variation operators on weighted Hardy and BMO spaces in the Schrödinger setting, *Bull. Malays. Math. Sci. Soc.* **45** no. 5 (2022), 2285–2312. DOI MR Zbl

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Received: November 1, 2022
Accepted: March 3, 2023