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ARITHMETIC PROPERTIES OF GENERALIZED FIBONACCI NUMBERS

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ABSTRACT. We present a survey of results concerning arithmetic properties of generalized Fibonacci sequences and certain Diophantine equations involving terms from that family of numbers. Most of these results have been recently obtained by the research groups in number theory at the Universities of Cauca (in Popayán) and of Valle (in Cali), Colombia, lead by the first two authors.

1. Generalized Fibonacci numbers

The Fibonacci sequence is one of the most famous and curious numerical sequences in mathematics and has been widely studied in the literature. Denoted by $\mathbf{F} := (F_n)_{n\geq 0}$, it has initial terms $F_0 = 0$, $F_1 = 1$ and obeys the recurrence $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. For the beauty and rich applications of these numbers and their relatives one can see Koshy's book [44].

The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. Here we consider, for an integer $k \geq 2$, the *k-Fibonacci sequence* $\mathbf{F}^{(k)} := (F_n^{(k)})_{n\geq -(k-2)}$ defined by the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all } n \geq 2, \tag{1.1}$$

with initial values $F_i^{(k)} = 0$ for $i = -(k-2), \ldots, 0$, and $F_1^{(k)} = 1$. We call $F_n^{(k)}$ the *n-th k-Fibonacci number*. The Fibonacci numbers are obtained for $k = 2$. When $k = 3$ the resulting sequence is sometimes called the *Tribonacci sequence* and is commonly denoted by $\mathbf{T} := (T_n)_n$.

The first $k+1$ non-zero terms of $\mathbf{F}^{(k)}$ are powers of 2, namely

$$F_1^{(k)} = 1 \quad \text{and} \quad F_n^{(k)} = 2^{n-2} \quad \text{for all } 2 \leq n \leq k+1,$$

while the next term is $F_{k+2}^{(k)} = 2^k - 1$. The inequality

$$F_n^{(k)} < 2^{n-2} \quad \text{holds for all } n \geq k+2$$

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(see [15]). Further, recursion (1.1) implies the three-term recursion

$$F_n^{(k)} = 2F_{n-1}^{(k)} - F_{n-k-1}^{(k)} \quad \text{for all } n \geq 3,$$

which also shows that $\mathbf{F}^{(k)}$ grows at a rate less than 2^{n-2}. In general, Howard and Cooper [39] proved the following nice formula.

Lemma 1.1. *For $k \geq 2$ and $n \geq k+2$,*

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} C_{n,j}\, 2^{n-(k+1)j-2},$$

where

$$C_{n,j} := (-1)^j \left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right].$$

In the above, we used the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of a or b is negative, and denoted by $\lfloor x \rfloor$ the greatest integer less than or equal to x. For example, assuming that $k+2 \leq n \leq 2k+2$, Howard and Cooper's formula becomes the identity

$$F_n^{(k)} = 2^{n-2} - (n-k) \cdot 2^{n-k-3}. \tag{1.2}$$

The classical study of linear recurrence sequences (see [30]) is based on knowledge of the roots of their characteristic polynomials. For an integer $k \geq 2$, the polynomial

$$\Psi_k(z) := z^k - z^{k-1} - \cdots - z - 1 \in \mathbb{Q}[z]$$

is the characteristic polynomial of $F^{(k)}$. While studying the roots of $\Psi_k(z)$ it is common to work with the polynomial

$$\psi_k(z) := (z-1)\Psi_k(z) = z^{k+1} - 2z^k + 1. \tag{1.3}$$

Except for the extra root at $z = 1$, $\psi_k(z)$ has the same roots as $\Psi_k(x)$. By Descartes' rule of signs, the polynomial $\Psi_k(z)$ has exactly one positive real root, say $z = \alpha$. Since $\Psi_k(1) = 1 - k$ and $\Psi_k(2) = 1$, it follows that $\alpha \in (1, 2)$. In fact, it is known that $2(1 - 2^{-k}) < \alpha < 2$ (see [40, Lemma 2.3] or [65, Lemma 3.6]). In addition, given that $\alpha^{k+1} - 2\alpha^k + 1 = 0$, we obtain $\alpha = 2 - \alpha^{-k} < 2 - 2^{-k}$ and so $2(1 - 2^{-k}) < \alpha < 2(1 - 2^{-(k+1)})$ for all $k \geq 2$. Thus, α approaches 2 at an exponential speed in k as k tends to infinity. Miles [53] showed that the roots of $\Psi_k(z)$ are distinct and the remaining $k-1$ roots of $\Psi_k(z)$ different from α lie inside the unit disk. He showed this by reducing the equation $\Psi_k(z) = 0$ to a form where Rouché's theorem could be applied. This fact was reproved by Miller [54] by an elementary argument. In particular, α is a Pisot number and $\Psi_k(z)$ is an irreducible polynomial over $\mathbb{Q}[z]$. It follows from a result of Mignotte [51] that if $\alpha_i \neq \alpha_j$ are roots of $\Psi_k(z)$ which are not complex conjugates, then $|\alpha_i| \neq |\alpha_j|$.

Gómez and Luca [33] proved a lower bound on the ratio of absolute values of roots of $\Psi_k(z)$. If α_i and α_j are roots of $\Psi_k(z)$ with $|\alpha_i| > |\alpha_j|$, they showed that

$$\frac{|\alpha_i|}{|\alpha_j|} > 1 + 8^{-k^4}.$$

This result was used in [36] as one of the main ingredients in the study of the zero-multiplicity of a particular linear recurrence sequence with characteristic polynomial $\Psi_k(z)$. In [33], Gómez and Luca gave upper and lower bounds on the absolute values of the roots of $\Psi(z)$ different from α. They proved that if α_i is a root of $\Psi_k(z)$ with $|\alpha_i| < 1$, then

$$1 - \frac{\log 3}{k} < |\alpha_i| < 1 - \frac{1}{2^8 k^3}.$$

Consider the function

$$f_k(z) := \frac{z-1}{2+(k+1)(z-2)}, \tag{1.4}$$

where $z \neq 2 - 2/(k+1)$. The inequalities

$$1/2 < f_k(\alpha) < 3/4 \qquad \text{and} \qquad |f_k(\alpha_i)| < 1, \quad 2 \leq i \leq k \tag{1.5}$$

hold, where $\alpha := \alpha_1, \ldots, \alpha_k$ are all the roots of $\Psi_k(z)$. Proofs for inequalities (1.5) can be found in [9]. With the above notation, Dresden and Du showed in [28] that the formula and the estimate

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i){\alpha_i}^{n-1} \qquad \text{and} \qquad \left| F_n^{(k)} - f_k(\alpha)\alpha^{n-1} \right| < \frac{1}{2}$$

hold for all $n \geq 1$ and $k \geq 2$. The equality from the left-hand side above is known as the Binet formula for $\mathbf{F}^{(k)}$. By the above relations, we can write

$$F_n^{(k)} = f_k(\alpha)\alpha^{n-1} + e_k(n),$$

where $|e_k(n)| < 1/2$ for all $n \geq 1$ and $k \geq 2$. A proof of the Binet formula for $\mathbf{F}^{(k)}$ can be given using generating functions. Below we present a different one based on the matrix method from [34] which is a consequence of the following more general result.

Theorem 1.2. *For the sequence* $\mathbf{U} := (U_n^{(k)})_{n\geq 0}$ *which satisfies the recurrence*

$$U_{n+k+1}^{(k)} = xU_{n+k}^{(k)} + yU_n^{(k)},$$

with the $k+1$ *initial values* $0, 1, x, \ldots, x^{k-1}$*, the Binet formula is given by*

$$U_n^{(k)} = \sum_{i=1}^{k+1} \frac{{\alpha_i}^n}{(k+1)\alpha_i - kx}, \tag{1.6}$$

where $\alpha_1, \ldots, \alpha_k, \alpha_{k+1}$ *are the roots of the polynomial* $f_{\mathbf{U}}(z) := z^{k+1} - xz^k - y$.

Proof. Let $\alpha_1, \ldots, \alpha_k, \alpha_{k+1}$ be the roots of the polynomial $z^{k+1} - xz^k - y$. It suffices to see that the n-th power of the $(k+1) \times (k+1)$ matrix

$$C := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ y & 0 & \cdots & 0 & x \end{pmatrix}$$

is given by

$$\begin{pmatrix} yu_{n-k}^{(k)} & yu_{n-k-1}^{(k)} & \cdots & yu_{n-2k+1}^{(k)} & u_{n-k+1}^{(k)} \\ yu_{n-k+1}^{(k)} & yu_{n-k}^{(k)} & \cdots & yu_{n-2k+2}^{(k)} & u_{n-k+2}^{(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ yu_{n-1}^{(k)} & yu_{n-2}^{(k)} & \cdots & yu_{n-k}^{(k)} & u_{n}^{(k)} \\ yu_{n}^{(k)} & yu_{n-1}^{(k)} & \cdots & yu_{n-k+1}^{(k)} & u_{n+1}^{(k)} \end{pmatrix},$$

and the Jordan canonical form of the matrix C is diagonal and given by $C = PDP^{-1}$, where

$$P := \begin{pmatrix} 1 & \dots & 1 & 1 \\ \alpha_1 & \dots & \alpha_k & \alpha_{k+1} \\ {\alpha_1}^2 & \dots & {\alpha_k}^2 & \alpha_{k+1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ {\alpha_1}^k & \cdots & {\alpha_k}^k & \alpha_{k+1}^k \end{pmatrix} \quad \text{and} \quad D := \begin{pmatrix} \alpha_1 & \dots & 0 & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \alpha_{k+1} \end{pmatrix}.$$

Setting $P^{-1} := (\beta_{ij})_{1\le i,j\le k+1}$, we have

$$\beta_{i,k+1} = \frac{1}{\prod_{j\neq i}(\alpha_i-\alpha_j)} = \frac{1}{(k+1)\alpha_i^k - kx\alpha_i^{k-1}}.$$

We now get the above equality (1.6) by identifying the scalar product of the $(k+1)$-st line of PD^n and the $k+1$-st column of P^{-1} with the term $U_{n+1}^{(r)}$. □

In particular, for the polynomial $\psi_k(z)$ defined in (1.3), we have that $x = 2$ and $y = -1$, so

$$U_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i^n}{2+(k+1)(\alpha_i-2)} + \frac{1}{1-k},$$

where $\alpha = \alpha_1, \dots, \alpha_k$, are the roots of the polynomial $\Psi_k(z)$. Now, we note that the sequence $U_n^{(k)} - U_{n-1}^{(k)}$ starts with the values

$$0,\ \ 1-0=1,\ \ 2-1=1,\ \ 4-2=2,\ \ \dots,\ \ 2^{k-1}-2^{k-2}=2^{k-2},$$

which are exactly the initial values of the sequence $\mathbf{F}^{(k)}$. Hence, this gives

$$\begin{aligned} F_n^{(k)} &= U_n^{(k)} - U_{n-1}^{(k)} \\ &= \sum_{i=1}^{k} \frac{\alpha_i^n}{2+(k+1)(\alpha_i-2)} - \sum_{i=1}^{k} \frac{\alpha_i^{n-1}}{2+(k+1)(\alpha_i-2)} \\ &= \sum_{i=1}^{k} \frac{\alpha_i - 1}{2+(k+1)(\alpha_i-2)}{\alpha_i}^{n-1}, \end{aligned}$$

which is the identity given by Dresden and Du. Regarding the growth of the sequence $\mathbf{F}^{(k)}$, Bravo and Luca [17] proved that the inequality

$$\alpha^{n-2} \le F_n^{(k)} \le \alpha^{n-1}$$

holds for all $n \geq 1$ and $k \geq 2$, exhibiting an exponential growth of the k-Fibonacci numbers as expected.

2. Intersections of linear recurrences

Many problems in number theory may be reduced to finding the intersection of two sequences of positive integers and it is for this reason that the problem of determining the intersection of two sequences has attracted attention from several number theorists. To fix the ideas, let $\mathbf{u} := (u_n)_{n\geq 0}$ and $\mathbf{v} := (v_n)_{n\geq 0}$ be linear recurrence sequences of integers. That is, there exist positive integers k, ℓ and integers $a_1, \dots, a_k, b_1, \dots, b_\ell$ such that both recurrences

$$\begin{aligned} u_{n+k} &= a_1 u_{n+k-1} + \cdots + a_k u_n, \\ v_{n+\ell} &= b_1 v_{n+\ell-1} + \cdots + b_\ell v_n \end{aligned} \tag{2.1}$$

hold for all $n \geq 0$. It is then well known that there exist distinct algebraic numbers $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_s$ and polynomials $P_1, \dots, P_r$ and $Q_1, \dots, Q_s$ such that the formulas

$$\begin{aligned} u_n &= P_1(n)\alpha_1^n + \cdots + P_r(n)\alpha_r^n, \\ v_n &= Q_1(n)\beta_1^n + \cdots + Q_s(n)\beta_s^n \end{aligned} \tag{2.2}$$

hold for all $n \geq 0$. If the recurrence relation (2.1) is *minimal*, that is, $\mathbf{u}$ does not satisfy a linear recurrence of order less than k (in particular, $a_k \neq 0$), then in the formula (2.2) the numbers $\alpha_1, \dots, \alpha_r$ are all the roots of the *characteristic polynomial*

$$P_{\mathbf{u}}(X) = X^k - a_1 X^{k-1} - \cdots - a_k = \prod_{i=1}^{k} (X - \alpha_i)^{\sigma_i},$$

and $P_i(X)$ are polynomials with complex coefficients of degree $\sigma_i - 1$, for $i = 1, \dots, k$, where σ_i is the multiplicity of α_i as a root of $P_{\mathbf{u}}(X)$. In practice, the coefficients of $P_i(X)$ for $i = 1, \dots, r$ can be found by using the initial values $u_0, \dots, u_{k-1}$ of $\mathbf{u}$. Similar considerations apply to $\mathbf{v}$. The linear recurrence sequence $\mathbf{u}$ is said to have a *dominant root* if, up to relabeling of the roots $\alpha_1, \dots, \alpha_r$, the inequality $|\alpha_1| > \max\{|\alpha_2|, \dots, |\alpha_r|\}$ holds. Note that necessarily $|\alpha_1| > 1$. In this case, α_1 is called the *dominant root* of $\mathbf{u}$. A question which has received considerable interest is to decide whether two linear recurrence sequences have only finitely or infinitely many common values. That is, whether the equation $u_n = v_m$ has only finitely many positive integer solutions (n, m). Below is a qualitative example due to Mignotte [50].

Theorem 2.1. *Assume that* $\mathbf{u}$ *and* $\mathbf{v}$ *are linear recurrence sequences whose general terms have representations as in* (2.2) *and which have dominant roots* α_1 *and* β_1*, respectively. There exists an effectively computable constant* n_0 *such that if* $u_n = v_m$ *holds with* $n \geq n_0$*, then* $P_1(n)\alpha_1^n = Q_1(m)\beta_1^m$*. If this last equation has infinitely many positive integer solutions* (n, m)*, then* α_1 *and* β_1 *are multiplicatively dependent; that is, there exist integers* x, y *not both zero such that* $\alpha_1^x = \beta_1^y$*.*

In the above statement and in what follows an effectively computable value n_0 means that one can write down a concrete bound for n_0 once the coefficients of the recurrences and the initial terms are given.

In light of Theorem 2.1 above, it follows that if $\mathbf{u}$ and $\mathbf{v}$ have dominant roots which are multiplicatively independent, then the equation $u_n = v_m$ has only finitely many positive integer solutions (n, m) and they are all effectively computable. To compute them, one uses Baker's method of linear forms in logarithms (see [3], for example) in the same way as Mignotte did. Let us give the main idea of his proof. Assume $u_n = v_m$ holds for some large value of $\max\{m, n\}$. The dominance conditions imply that

$$|u_n| \asymp n^d |\alpha_1|^n, \quad |v_m| \asymp m^e |\beta_1|^m,$$

where d, e are the degrees of $P_1(z)$ and $Q_1(z)$, respectively. Hence,

$$|n \log |\alpha_1| - m \log |\beta_1|| = O(\log n),$$

so n and m are almost proportional as $\max\{m, n\}$ becomes large. Thus, both become large if one of them does. Assume that both m, n are larger than the maximal real root of $P_1(z)Q_1(z)$. We rewrite

$$u_n = v_m \quad \text{as} \quad |P_1(n)\alpha_1^n - Q_1(m)\beta^m| = \left| \sum_{j=2}^{s} Q_j(m)\beta_j^m - \sum_{i=2}^{r} P_i(n)\alpha_i^n \right|,$$

and deduce that

$$\left|1 - P_1(n)^{-1} Q_1(m) \alpha_1^{-n} \beta_1^m\right| \ll \frac{n^D}{\rho^n}, \tag{2.3}$$

where $D := \max\{\deg P_1(z), \deg Q_1(z)\}$ and $\rho := \max_{i,j}\{|\alpha_1|/|\alpha_i|, |\beta_1|/|\beta_j|\}$. If the left-hand side above is non-zero, then Baker's bound together with the fact that m and n are proportional implies that the left-hand side is bounded below by $\exp(-C_{\mathbf{u},\mathbf{v}}(\log n)^3)$ with some constant $C_{\mathbf{u},\mathbf{v}}$ depending only on $P_1(z), Q_1(z), \alpha_1, \beta_1$, which together with (2.3) bounds n. This is all there is to it. Thus, given $\mathbf{u}$ and $\mathbf{v}$ with the above conditions (that they have dominant roots which are multiplicatively independent), Baker's method produces a numerical bound on $\max\{n, m\}$ over all the positive integer solutions (n, m) of the Diophantine equation $u_n = v_m$. In practice, these bounds are huge because $C_{\mathbf{u},\mathbf{v}}$ is large, and they need to be reduced using techniques from continued fractions or the LLL algorithm (see sections 2.3.3 and 2.3.4 of Cohen's book [24]) in order to bring them to small enough values where one can find the solutions by simply enumerating them in the remaining small range. It is interesting to bound at the theoretical level the number of solutions $u_n = v_m$, or to at least show that it has very few "large" solutions. This was done recently by Bennett and Pintér [5]. To state their result, we need one additional definition. For an algebraic number α of minimal polynomial

$$a_0 X^d + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \alpha_i) \in \mathbb{Z}[X],$$

where $a_0 > 0$, define the *height* of α as

$$h(\alpha) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\alpha^{(i)}|, 1\}\right).$$

The following theorem is the result of Bennett and Pintér's paper [5].

Theorem 2.2. *Assume that* $\mathbf{u}$ *and* $\mathbf{v}$ *are given linear recurrence sequences with dominant roots* α_1 *and* β_1 *which are multiplicatively independent, whose general terms are given by* (2.2) *with nonzero algebraic numbers* $P_1, \ldots, P_r$ *and* $Q_1, \ldots, Q_s$*. Assume further that* $P_1 \neq Q_1$*. Put*

$$\begin{aligned} M &:= \max\{h(P_i), h(Q_j) : 1 \le i \le r,\ 1 \le j \le s\}, \\ N &:= \max\{r, s, \log|\beta_1|, 3\}. \end{aligned} \tag{2.4}$$

Then there exists an effectively computable constant C *such that if*

$$\log|\alpha_1| > CM \log|\beta_1| \log^3 N, \tag{2.5}$$

then there is at most one pair of positive integers (n, m) *such that* $u_n = v_m$ *and* $P_1\alpha_1^n = Q_1\beta_1^m$*.*

The above theorem has the advantage that it guarantees that the Diophantine equation $u_n = v_m$ has at most one large solution when only one of the sequences, say $\mathbf{v}$, is fully known while we only have some clues about the second one, $\mathbf{u}$. That is, assume that $\mathbf{v}$ is given. Assume also that $\mathbf{u}$ is not given, but that we know r and we know bounds on the heights of the coefficients $P_1, \ldots, P_r$. Then M and N in (2.4) are determined. What we are missing to fully know $\mathbf{u}$ are the roots $\alpha_1, \ldots, \alpha_r$. Then Theorem 2.2 says that if $|\alpha_1|$ is large enough such that inequality (2.5) is satisfied, then the equation $u_n = v_m$ has at most one positive integer solution (n, m).

We will indicate some practical examples of these results in the next sections.

3. Examples with repdigits and generalized Fibonacci sequences

A repdigit is a positive integer whose base 10-representation has one repeated digit. Thus, a repdigit is a positive integer of the form

$$N = a\left(\frac{10^m - 1}{9}\right) \quad \text{for some } a \in \{1, \ldots, 9\} \text{ and some } m \ge 1.$$

Many papers have been written on Diophantine equations involving repdigits and terms of certain linear recurrence sequences. For example, Luca [45] showed that 55 and 11 are the largest repdigits in the Fibonacci and Lucas sequences, respectively. Since then, this result has been generalized and extended in various directions. Erduvan and Keskin [29] found all repdigits expressible as products of two Fibonacci or Lucas numbers. Faye and Luca [31] looked for repdigits in the usual Pell sequence and using some elementary methods they concluded that there are no Pell numbers larger than 10 that are repdigits. Normenyo, Luca, and Togbé found all repdigits expressible as sums of three Pell numbers [58] and later four Pell numbers [46].

For linear recurrences of higher order Marques [48] proved that 44 is the largest repdigit in the Tribonacci sequence. Moreover, Marques conjectured that there are no repdigits with at least two digits belonging to $\mathbf{F}^{(k)}$ for any $k > 3$.

The sequence $\mathbf{v}$ of repdigits is not a linear recurrence sequence but it is the union of 9 linear recurrence sequences by fixing the value of $a \in \{1, \ldots, 9\}$. They all satisfy formula (2.2) with $s = 2$, $(\beta_1, \beta_2) = (10, 1)$, and $(Q_1, Q_2) = (a/9, -a/9)$. In [17], Bravo and Luca have treated the problem of determining all repdigits in $\mathbf{F}^{(k)}$ with the aim of confirming Marques' conjecture. They considered the Diophantine equation

$$F_n^{(k)} = a\left(\frac{10^\ell - 1}{9}\right)$$

in positive integers n, k, ℓ, a with $k \geq 2$, $\ell \geq 2$, and $a \in \{1, 2, \ldots, 9\}$ and proved the following theorem about it.

Theorem 3.1. *If $F_n^{(k)}$ is a repdigit with at least two digits, then $(n, k) = (10, 2)$ or $(8, 3)$. Namely, the only examples are $F_{10} = 55$ and $T_8 = 44$.*

One of the main ingredients to prove Theorem 3.1 was lower bounds for linear forms in logarithms of algebraic numbers to bound n polynomially in terms of k. When k is small, the theory of continued fractions sufficed to lower such bounds to reasonable small ranges and complete the calculations by brute force. When k is large, they used the fact that the dominant root of $\mathbf{F}^{(k)}$ denoted by α in the previous section is exponentially close to 2, so they could replace this root by 2 in their calculations with linear forms in logarithms and end up with an absolute bound for the variables, which can be reduced by using again standard facts concerning continued fractions. The following estimate due to Bravo and Luca [17] was an important tool in addressing the large values of k.

Lemma 3.2. *For $k \geq 2$, let $\alpha = \alpha(k)$ be the dominant root of $\mathbf{F}^{(k)}$, and consider the function $f_k(x)$ defined in (1.4). Suppose that $n > 1$ is an integer satisfying $n < 2^{k/2}$, and put $\delta = \alpha^{n-1} - 2^{n-1}$ and $\eta = f_k(\alpha) - f_k(2)$. Then*

$$|\delta| < \frac{2^n}{2^{k/2}} \quad \text{and} \quad |\eta| < \frac{2k}{2^k}.$$

Moreover,

$$f_k(\alpha)\alpha^{n-1} = 2^{n-2} + \frac{\delta}{2} + 2^{n-1}\eta + \eta\delta.$$

Alahmadi et al. [1] generalized the result given in Theorem 3.1 by showing that the only repdigits with at least two digits which can be represented as a product of ℓ consecutive k-Fibonacci numbers (here, $\ell \geq 1$) occur only for $(k, \ell) = (2, 1), (3, 1)$, in such a way extending the works [7, 49] which dealt with the particular cases of Fibonacci and Tribonacci numbers.

We finish this section by mentioning that additive variants of Theorem 3.1 have also been considered. For example, paper [19] extended the previous work from [17] and searched for repdigits which are sums of two k-Fibonacci numbers; i.e., they determined all the solutions of the Diophantine equation

$$F_n^{(k)} + F_m^{(k)} = a\left(\frac{10^\ell - 1}{9}\right) \tag{3.1}$$

in integers n, m, k, a and ℓ with $k \geq 2$, $n \geq m$, $1 \leq a \leq 9$, and $\ell \geq 2$. Before presenting the next result, we observe that in equation (3.1) above one can assume $m \geq 1$, since otherwise $F_m^{(k)} = 0$ and therefore the resulting expression would be an equation which was completely solved in [17]. The following result was obtained in [19].

Theorem 3.3. *All solutions of the Diophantine equation* (3.1) *with* $n \geq m \geq 1$, $k \geq 2$, $\ell \geq 2$, *and* $a \in \{1, \dots, 9\}$ *are:*

$$\begin{array}{lll}
F_6 + F_4 = 11 & F_9^{(3)} + F_5^{(3)} = 88 & F_8^{(5)} + F_6^{(5)} = 77 \\
F_8 + F_i = 22,\ i = 1, 2 & F_7^{(4)} + F_4^{(4)} = 33 & F_{11}^{(6)} + F_8^{(6)} = 555 \\
F_9 + F_8 = 55 & F_7^{(4)} + F_6^{(4)} = 44 & F_7^{(k)} + F_i^{(k)} = 33,\ \forall k \geq 6,\ i = 1, 2 \\
2F_8^{(3)} = 88 & F_{12}^{(4)} + F_4^{(4)} = 777 & F_8^{(k)} + F_3^{(k)} = 66,\ \forall k \geq 7 \\
F_5^{(3)} + F_4^{(3)} = 11 & F_7^{(5)} + F_3^{(5)} = 33. &
\end{array}$$

The following estimate due to Bravo, Gómez, and Luca [10] is an improvement on Lemma 3.2 and it is currently one of the key points in addressing the large values of k when solving Diophantine equations involving terms of generalized Fibonacci numbers (see also [8]).

Lemma 3.4. *Let* $k \geq 2$ *and suppose that* $n < 2^{k/2}$. *Then*

$$F_n^{(k)} = 2^{n-2}(1 + \zeta_f), \quad \text{where } |\zeta_f| < \frac{1}{2^{k/2}}.$$

Proof. It follows by using the Howard and Cooper formula given in Lemma 1.1. □

In [27], Díaz and Luca found all Fibonacci numbers as sums of two repdigits. One may then study an analogue of the problem of Díaz and Luca when the sequence of Fibonacci numbers is replaced by the sequence of k-Fibonacci numbers. In [10], we considered this variant and by using Lemma 3.4 to deal with the large values of k we obtained the following result.

Theorem 3.5. *For* $k \geq 3$ *and* $n \geq k + 2$, *the Diophantine equation*

$$F_n^{(k)} = a\left(\frac{10^m - 1}{9}\right) + b\left(\frac{10^\ell - 1}{9}\right), \quad 1 \leq a,\ b \leq 9, \tag{3.2}$$

has only the following 17 *positive integer solutions* (n,k,m,ℓ,a,b)*,* $m \geq \max\{\ell,2\}$*:*

$$\begin{array}{lll}
F_6^{(3)} = 13 = 11+2 & F_7^{(3)} = 24 = 22+2 & F_9^{(3)} = 81 = 77+4 \\
F_6^{(4)} = 15 = 11+4 & F_7^{(4)} = 29 = 22+7 & F_8^{(4)} = 56 = 55+1 \\
F_9^{(4)} = 108 = 99+9 & F_7^{(5)} = 31 = 22+9 & F_8^{(5)} = 61 = 55+6 \\
F_9^{(5)} = 120 = 111+9 & F_8^{(6)} = 63 = 55+8 & F_{12}^{(7)} = 1004 = 999+5 \\
F_8^{(3)} = 44 = 11+33 & F_8^{(3)} = 44 = 22+22 & F_{12}^{(6)} = 976 = 888+88 \\
F_{10}^{(8)} = 255 = 222+33 & F_{12}^{(9)} = 1021 = 999+22. &
\end{array}$$

On the other hand, for $n < k+2$*,* $F_n^{(k)} = 2^{n-2}$ *and the only solutions of* (3.2) *with* $m \geq \max\{\ell,2\}$ *are*

$$F_6^{(k)} = 16 = 11+5 \quad (k \geq 5) \qquad \textit{and} \qquad F_8^{(k)} = 64 = 55+9 \quad (k \geq 7).$$

We clarify that the condition $m \geq \max\{\ell,2\} \geq 2$, in the above theorem is only meant to insure that $F_n^{(k)}$ has at least 2 digits and so to avoid small numbers which are sums of two one-digit numbers.

4. Coincidences in generalized Fibonacci sequences

In 2005, Noe and Post [57] proposed a conjecture about coincidences of terms of generalized Fibonacci sequences. In their work, they gave a heuristic argument to show that if $k \neq \ell$, then the cardinality of the intersection $F^{(k)} \cap F^{(\ell)}$ must be small. Further, they used computational methods which led them to confirm their conjecture for all terms of size less than 22000. This conjecture has been proved to hold independently by Bravo and Luca [16] and by Marques [47]. They considered the Diophantine equation

$$F_n^{(k)} = F_m^{(\ell)} \tag{4.1}$$

in positive integers n,k,m,ℓ with $k > \ell \geq 2$.

In light of the results and remarks from Section 2, one needs to prove that the dominant roots of $\mathbf{u} = F^{(k)}$ and $\mathbf{v} = F^{(\ell)}$ are multiplicatively independent, which implies that the intersection $F^{(k)} \cap F^{(\ell)}$ is finite. Indeed, let α and β be the dominant roots of $\mathbf{u} = F^{(k)}$ and $\mathbf{v} = F^{(\ell)}$, respectively, and suppose that $\alpha^x = \beta^y$ for some integers x,y not both zero. Up to replacing (x,y) by $(-x,-y)$ we may assume that $y \geq 0$. Since $|\alpha| > 1$, $|\beta| > 1$, we get that $x \geq 0$. So, in fact, we may assume that both x and y are positive since one of them being zero entails that the other is zero as well, which is not possible. Let $\mathbb{L}$ be the normal closure of $\mathbb{K} = \mathbb{Q}(\alpha,\beta)$ and let further $\sigma_1,\ldots,\sigma_k$ be elements of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ such that $\sigma_i(\alpha) = \alpha_i$. Since $k > \ell$, there exist $i \neq j$ in $\{1,2,\ldots,k\}$ such that $\sigma_i(\beta) = \sigma_j(\beta)$. Hence, the automorphism $\sigma = \sigma_j^{-1}\sigma_i$ satisfies $\sigma(\beta) = \beta$ and $\sigma(\alpha) = \alpha_s$, where $s \neq 1$ is such that $\sigma_j^{-1}(\alpha_i) = \alpha_s$. Applying σ to the above relation and taking absolute values we get that $|\alpha_s|^x = \beta^y$, which is not possible because $|\alpha_s| < 1$. Thus, α and β are multiplicatively independent.

Note next that since the first $k+1$ non-zero terms in $F^{(k)}$ are powers of 2, we have that $F_t^{(k)} = F_t^{(\ell)}$ for all $1 \le t \le \ell + 1$; i.e., the quadruple

$$(n, k, m, \ell) = (t, k, t, \ell) \tag{4.2}$$

is a solution of equation (4.1) for all $1 \le t \le \ell + 1$. The solutions shown at (4.2) will be called *trivial solutions.* The following result was obtained in [16].

Theorem 4.1. *The only nontrivial solutions of the Diophantine equation* (4.1) *in positive integers* n, k, m, ℓ *with* $k > \ell \ge 2$, *are*

$$(n, k, m, \ell) \in \{(6, 3, 7, 2), (11, 7, 12, 3), (6, 2, 5, t)\}$$

for all $t \ge 4$. *Namely,* $F_6^{(3)} = F_7^{(2)} = 13$, $F_{11}^{(7)} = F_{12}^{(3)} = 504$, *and* $F_5^{(t)} = F_6^{(2)} = 8$ *for all* $t \ge 4$.

A similar program for k-Lucas sequences was performed in [6], where the k-Lucas sequence follows the same recurrence relation as the k-Fibonacci numbers except that it starts with the k-tuple of initial values $0, \dots, 0, 2, 1$.

5. On the largest prime factor of generalized Fibonacci numbers

For an integer m, let $P(m)$ denote the largest prime factor of m with the convention that $P(0) = P(\pm 1) = 1$. The problem of finding lower bounds for the largest prime factor of terms of linear recurrence sequences has attracted the attention of several number theorists. There are many papers in the literature with interesting results about this problem. For example, Schinzel [62] showed that $P(2^n - 1) \ge 2n + 1$ for all $n > 12$. For large $n > n_0$, Stewart [63] did much better and proved that

$$P(2^n - 1) > n \exp(\log n / (104 \log\log n)), \tag{5.1}$$

confirming a conjecture of Erdős to the effect that $\lim_{n\to\infty} P(2^n - 1)/n = \infty$. Stewart did not estimate n_0. Under the *abc*-conjecture, Murty and Wong [56] proved that $P(2^n - 1) > n^{2-\varepsilon}$ holds for all $\varepsilon > 0$ once n is sufficiently large in a way depending on ε. Murata and Pomerance [55] wrote: “It is perhaps reasonable to conjecture that $P(2^n - 1) > n^K$ for all sufficiently large K and all sufficiently large n depending on K, or maybe even $P(2^n - 1) > 2^{n/\log n}$ for all sufficiently large n, but clearly we are very far from proving such assertions”.

A similar approach was followed for the k-Fibonacci sequence and an effective lower bound for $P(F_n^{(k)})$ in terms of both the parameters k and n was obtained by Bravo and Luca in [18].

Theorem 5.1. *The inequality*

$$P(F_n^{(k)}) > 0.01\sqrt{\log n \log\log n}$$

holds for all $k \ge 2$ *and* $n \ge k + 2$.

Note that the condition $n \ge k + 2$ above is needed just because $F_n^{(k)}$ is a power of 2 for all $n \in [1, k+1]$ so $P(F_n^{(k)}) = 1, 2$ in such cases. For the Fibonacci sequence, Carmichael's Primitive Divisor theorem [22] states that for $n > 12$, the

n-th Fibonacci number F_n has at least one prime factor that is not a factor of any previous Fibonacci number. Such prime factors are called *primitive* for F_n. The primitive prime factors of F_n are congruent to $\pm 1 \pmod{n}$ so, in particular, they are at least as large as $n-1$; hence, for the Fibonacci numbers we have that $P(F_n) \geq n-1$ whenever $n \geq 13$, which is a much better result than the one given in Theorem 5.1. In fact, Stewart's inequality (5.1) holds with $2^n - 1$ replaced by F_n, again for all $n > n_0$, where n_0 is a number which Stewart did not compute. In the general case, that is, if we replace F_n by $F_n^{(k)}$ for some $k \geq 3$, there is no result comparable to either Carmichael's or Stewart's theorems.

In [18], the authors used the LLL algorithm and proved a numerical result by finding all the k-Fibonacci numbers whose largest prime factor is less than or equal to 7, i.e., they determined all the solutions of the Diophantine equation

$$F_n^{(k)} = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \tag{5.2}$$

in nonnegative integers n, k, a, b, c, d with $k \geq 2$. Note again that it suffices to consider the case when $n \geq k+2$, otherwise (5.2) holds trivially, since the first $k+1$ nonzero terms in $F^{(k)}$ are powers of 2.

Theorem 5.2. *The only nontrivial solutions of the Diophantine equation* (5.2) *are:*

$$\begin{array}{lll}
F_4^{(2)} = 3 & F_9^{(3)} = 81 = 3^4 & F_8^{(6)} = 63 = 3^2 \cdot 7 \\
F_5^{(2)} = 5 & F_{12}^{(3)} = 504 = 2^3 \cdot 3^2 \cdot 7 & F_9^{(6)} = 125 = 5^3 \\
F_6^{(2)} = 8 = 2^3 & F_{15}^{(3)} = 3136 = 2^6 \cdot 7^2 & F_{14}^{(6)} = 3840 = 2^8 \cdot 3 \cdot 5 \\
F_8^{(2)} = 21 = 3 \cdot 7 & F_6^{(4)} = 15 = 3 \cdot 5 & F_{11}^{(7)} = 504 = 2^3 \cdot 3^2 \cdot 7 \\
F_{12}^{(2)} = 144 = 2^4 \cdot 3^2 & F_8^{(4)} = 56 = 2^3 \cdot 7 & F_{13}^{(7)} = 2000 = 2^4 \cdot 5^3 \\
F_5^{(3)} = 7 & F_9^{(4)} = 108 = 2^2 \cdot 3^3 & F_{16}^{(8)} = 16128 = 2^8 \cdot 3^2 \cdot 7 \\
F_7^{(3)} = 24 = 2^3 \cdot 3 & F_9^{(5)} = 120 = 2^3 \cdot 3 \cdot 5. &
\end{array}$$

A more general problem was studied by Gómez and Luca in [34]. Namely, for positive integers n, m, $k \geq 2$, $\ell \geq 2$ we write

$$\frac{F_n^{(k)}}{F_m^{(\ell)}} = \frac{a}{b} \tag{5.3}$$

in reduced form, that is, $\gcd(a, b) = 1$. We extend the largest prime factor function to rational numbers a/b in reduced form by putting $P(a/b) := P(ab) = \max\{P(a), P(b)\}$.

Let T be a fixed parameter and (n, k, m, ℓ) be a quadruple of positive integers for which $P(a/b) \leq T$, where a/b is given by (5.3). Gómez and Luca went on to show that this inequality has only finitely many solutions which are *nontrivial* in a suitable sense. The case $T = 1$ reduces to equation (4.1), studied in [16]. So, assume that $T \geq 2$. If $T = 2$ and $k = \ell$, then one encounters the Diophantine equation $F_n^{(k)} = 2^s F_m^{(k)}$ studied in [32] and its only solutions have either $s = 0$ (so, $m = n$), or $F_m^{(k)}$ is a power of 2 (so also $F_n^{(k)}$ is a power of 2). If T is arbitrary but say $m \leq \ell+1$, we then get that $F_m^{(\ell)}$ is a power of 2, so $P(F_n^{(k)}) \leq T$, and assuming that $n \geq k+2$, we get that $T \geq 0.01\sqrt{\log n \log\log n}$ by Theorem 5.1. A similar argument applies if $n \leq k+1$.

So, the problem becomes interesting whenever $n \geq k+2$, $m \geq \ell+2$, and $(n,k) \neq (m,\ell)$ in (5.3). Assuming that $k+2 \leq n \leq 2k+2$ and $\ell+2 \leq m \leq 2\ell+2$, by identity (1.2), $F_n^{(k)} = 2^{n-2} - (n-k)2^{n-k-3}$ and $F_m^{(\ell)} = 2^{m-2} - (m-\ell)2^{m-\ell-3}$. If further $a/b = 2^{n-m}$, the equation

$$\frac{F_n^{(k)}}{F_m^{(\ell)}} = \frac{a}{b} = 2^{n-m}$$

becomes

$$2^{n-2} - (n-k)2^{n-k-3} = 2^{n-m}(2^{m-2} - (m-\ell)2^{m-\ell-3}),$$

which is equivalent to

$$n - k = 2^{k-\ell}(m-\ell).$$

The above leads to infinitely many examples. For example, fix $m - \ell = s \geq 2$ and $k - \ell = t$ such that $2^t s \geq 2$ is an integer (t may be negative as long as $2^t s$ is an integer). This is fulfilled assuming that $t \geq \max\{-\nu_2(s), 1 - \log s/\log 2\}$, where $\nu_2(m)$ is the exponent of 2 in the factorization of m. Then $k = \ell + t$, $m = \ell + s$, and $n = k + 2^t s = \ell + (t + 2^t s)$. The only additional conditions now to be satisfied are that $m = \ell + s \leq 2\ell + 2$ and $n = \ell + (t + 2^t s) \leq 2\ell + 2s + 2 = 2k + 2$, which hold provided that

$$\ell \geq \max\{s-2, 2^t s + t - 2s - 2\},$$

as well as $(n,k) \neq (m,\ell)$, which is equivalent to $t \neq 0$. The main result in [34] is that $P(a/b)$ tends to infinity with n uniformly in k, ℓ, m except for the above situation.

Theorem 5.3. *If $n \geq k+2$, $m \geq \ell+2$, $n \geq m$, $(n,k) \neq (m,\ell)$, then the inequality*

$$P\left(\frac{F_n^{(k)}}{F_m^{(\ell)}}\right) > (\log n)^{1/5}$$

holds for all $n > 10^{10^9}$ except when $n \leq 2k+2$, $m \leq 2\ell+2$, and $a/b = 2^{n-m}$, which is the situation described above. The exponent $1/5$ can be replaced by $1/4-\varepsilon$ for any $\varepsilon > 0$ provided $n > n_0(\varepsilon)$, where this last number is a constant which can be effectively computed in terms of ε.

Instead of looking at the largest prime factor of a ratio of two k-Fibonacci numbers, one can look at prime factors of sums of k-Fibonacci numbers. Bravo and Luca [20] solved the Diophantine equation

$$F_n + F_m = 2^a, \qquad \text{with } n \geq m \geq 2 \text{ and } a \geq 1, \tag{5.4}$$

showing that its only solutions are $(n, m, a) = (4, 2, 2)$, $(5, 4, 3)$ and $(7, 4, 4)$. Motivated by their paper, Pink and Ziegler [61] fixed a non-degenerate binary recurrence sequence $(u_n)_{n\geq 0}$ and studied the Diophantine equation

$$u_n + u_m = wp_1^{z_1} \cdots p_s^{z_s} \qquad \text{for } n \geq m,$$

where w is a fixed non-zero integer and $p_1, p_2, \ldots, p_s$ are fixed distinct prime numbers. The unknowns are the positive integers m and n and the nonnegative exponents $z_1, \ldots, z_s$. Under mild technical restrictions they proved an effective finiteness result for the solutions of the above equation. For the particular case of the Fibonacci sequence they studied the numerical case

$$F_n + F_m = 2^{z_1} \cdot 3^{z_2} \cdots 199^{z_{46}} \tag{5.5}$$

in nonnegative integer unknowns $n, m, z_1, \ldots, z_{46}$ with $n \geq m$, showing there are exactly 325 solutions $(n, m, z_1, \ldots, z_{46})$. All of them have $n \leq 59$.

The three of us investigated the analogous equation to (5.4) when the sequence of Fibonacci numbers is replaced by the sequence of k-Fibonacci numbers. To be more precise, we considered the Diophantine equation

$$F_n^{(k)} + F_m^{(k)} = 2^a \tag{5.6}$$

in positive integers n, m, k and a with $k \geq 3$ and $n \geq m$. The complete list of solutions of (5.6) appears in [9]. Here is that result.

Theorem 5.4. *Let (n, m, k, a) be a solution of the Diophantine equation* (5.6)*. If $n = m$, then $(n, m, a) = (t, t, t-1)$ for all $2 \leq t \leq k+1$ or $(n, m, a) = (1, 1, 1)$. If $n > m$ and $a \neq n-2$, then the only solution is $(n, m, a) = (2, 1, 1)$, while if $n > m$ and $a = n - 2$, then all the solutions are given by*

$$(n, m, a) = (k + 2^\ell, 2^\ell + \ell - 1, k + 2^\ell - 2), \tag{5.7}$$

where ℓ is a positive integer such that $2^\ell + \ell - 2 \leq k$. In particular, we have $m \leq k+1$ and $n \leq 2k+1$.

Inspired by the work of Pink and Ziegler [61], Gómez and Luca considered in [35] an extension of the Diophantine equations (5.4) and (5.6) to the case where the right-hand sides are replaced by S-integers instead of powers of 2. Here, S-integers are integers whose prime factors belong to a fixed predetermined finite set of prime numbers denoted by S. Thus, they studied the growth of $P(F_n^{(k)} + F_m^{(k)})$ obtaining the following result.

Theorem 5.5. *The inequality*

$$P\big(F_n^{(k)} + F_m^{(k)}\big) > \frac{1}{200}\sqrt{\log n \log\log n}$$

holds for all $n \geq m$, $n \geq k+2$, and $k \geq 2$ except when $k + 2 \leq n \leq 2k+2$ and $m \leq k+2$ are part of the solutions to (5.6) *of the form* (5.7) *described in Theorem 5.4 (for some ℓ).*

A consequence of Theorem 5.5 is that given a finite set of primes, say $S = \{p_1, \ldots, p_s\}$, the S-integers which can be written as a sum of two k-Fibonacci numbers, where k is also unknown, form a finite effectively computable set. As an example, they found all sums of two k-Fibonacci numbers whose largest prime factor is less than or equal to 7. That is,

$$F_n^{(k)} + F_m^{(k)} = 2^a \cdot 3^b \cdot 5^c \cdot 7^d, \tag{5.8}$$

with n, m, k, a, b, c, d being nonnegative integers with $n > m \geq 2$, $k \geq 2$. The case $k = 2$ is a particular case of the more general equation (5.5) solved by Pink and Ziegler [61]. These solutions are:

$F_3 + F_2 = 3$	$F_4 + F_2 = 2^2$	$F_5 + F_2 = 2 \cdot 3$
$F_6 + F_2 = 3^2$	$F_7 + F_2 = 2 \cdot 7$	$F_9 + F_2 = 5 \cdot 7$
$F_{10} + F_2 = 2^3 \cdot 7$	$F_{11} + F_2 = 2 \cdot 3^2 \cdot 5$	$F_{14} + F_2 = 2 \cdot 3^3 \cdot 7$
$F_4 + F_3 = 5$	$F_5 + F_3 = 7$	$F_6 + F_3 = 2 \cdot 5$
$F_7 + F_3 = 3 \cdot 5$	$F_9 + F_3 = 2^2 \cdot 3^2$	$F_5 + F_4 = 2^3$
$F_7 + F_4 = 2^4$	$F_8 + F_4 = 2^3 \cdot 3$	$F_{12} + F_4 = 3 \cdot 7^2$
$F_{17} + F_4 = 2^6 \cdot 5^2$	$F_5 + F_7 = 2 \cdot 3^2$	$F_{10} + F_5 = 2^2 \cdot 3 \cdot 5$
$F_7 + F_6 = 3 \cdot 7$	$F_9 + F_6 = 2 \cdot 3 \cdot 7$	$F_{10} + F_6 = 3^2 \cdot 7$
$F_{18} + F_6 = 2^5 \cdot 3^4$	$F_{16} + F_7 = 2^3 \cdot 5^3$	$F_{16} + F_8 = 2^4 \cdot 3^2 \cdot 7$
$F_{11} + F_{10} = 2^4 \cdot 3^2$	$F_{13} + F_{10} = 2^5 \cdot 3^2$	$F_{14} + F_{10} = 2^4 \cdot 3^3.$

Completing the table above to include the solutions with $k \geq 3$ yields the following statement.

Theorem 5.6. *Let (n, m, k, a, b, c, d) be a solution of Diophantine equation (5.8) with $n > m \geq 2$, $k \geq 3$, and $bcd \neq 0$. If $n \leq k + 1$, then $n - m \in \{1, 2, 3\}$. Otherwise,*

$$k \leq 320 \quad \textit{and} \quad \max\{n, m, a, b, c, d\} \leq 775.$$

More exactly, the equation has

(i) *34 solutions when $m \leq k + 1$ and $k + 2 \leq n \leq 2k + 1$;*
(ii) *7 solutions when $m \leq k + 1$ and $n \geq 2k + 2$;*
(iii) *14 solutions when $k + 2 \leq m \leq n$.*

6. Pillai's problem with generalized Fibonacci numbers

Suppose that a, b, and c are fixed nonzero integers and consider the exponential Diophantine equation

$$a^x - b^y = c. \tag{6.1}$$

In 1936 and again in 1945 (see [59]), Pillai formulated and recalled his famous conjecture which states that for any fixed nonzero integer c, the Diophantine equation (6.1) has only finitely many positive integer solutions (a, b, x, y) with $x, y \geq 2$. This conjecture is still open for all $c \neq \pm 1$. The case $c = \pm 1$ is Catalan's conjecture, solved by Mihăilescu [52].

The work started by Pillai was continued in 1936 by A. Herschfeld [37, 38], who proved that equation (6.1) has finitely many solutions in the particular case $(a, b) = (2, 3)$. Pillai [59, 60], extended Herschfeld's result to general (a, b) with $\gcd(a, b) = 1$ and $a > b \geq 2$. Specifically, Pillai showed that there exists a positive integer $c_0(a, b)$ such that, for $|c| > c_0(a, b)$, equation (6.1) has at most one positive integer solution (x, y). In particular, he conjectured that if $(a, b) = (2, 3)$ and $|c| > 13$, then equation (6.1) has at most one solution. This conjecture was confirmed by Stroeker and Tijdeman [64] and their result was further improved by Bennett [4], who showed that equation (6.1) has at most two solutions for fixed a, b, and c with $a, b > 2$.

Some recent results related to equation (6.1) have been obtained by several authors in the context of linear recurrence sequences, i.e., by replacing the powers of a and b by members of linear recurrence sequences. To fix ideas, let $\mathbf{u} := (u_n)_{n\geq 0}$ and $\mathbf{v} := (v_n)_{n\geq 0}$ be two linear recurrence sequences of integers and consider the Diophantine equation

$$u_n - v_m = c \tag{6.2}$$

for a fixed integer c and positive integers n and m. Chim, Pink, and Ziegler [23] studied equation (6.2) obtaining that if $\mathbf{u}$ and $\mathbf{v}$ have dominant roots which are multiplicatively independent, and $m_{\mathbf{u},\mathbf{v}}(c)$ denotes the multiplicity of c as an element of the form $u_n - v_m$, while $\mathcal{C}_{\mathbf{u},\mathbf{v}} := \{c \in \mathbb{Z} : m_{\mathbf{u},\mathbf{v}}(c) \geq 2\}$ represents the set of exceptions for the Pillai equation corresponding to the pair of sequences $(\mathbf{u}, \mathbf{v})$, then $\mathcal{C}_{\mathbf{u},\mathbf{v}}$ is finite and effectively computable. That is, there is an integer $c_0(\mathbf{u}, \mathbf{v}) > 0$ such that $m_{\mathbf{u},\mathbf{v}}(c) \leq 1$ for all $|c| > c_0(\mathbf{u}, \mathbf{v})$.

This variant was started by Ddamulira, Luca, and Rakotomalala [26] with Fibonacci numbers and powers of 2. They proved that the only integers c having at least two representations of the form $F_n - 2^m$ are contained in the set $\mathcal{C} = \{0, -1, 1, -3, 5, -11, -30, 85\}$. Shortly afterwards, Bravo, Luca, and Yazán [21] considered the same Diophantine equation in Tribonacci numbers instead of Fibonacci numbers. In their paper, they proved that the only integers c having at least two representations of the form $T_n - 2^m = c$ are contained in the set $\mathcal{C} = \{0, -1, -3, 5, -8\}$. In order to extend this problem to generalized Fibonacci numbers, Ddamulira, Gómez, and Luca [25] investigated those c admitting at least two representations of the form $F_n^{(k)} - 2^m$ for some positive integers k, n and m. This can be interpreted as solving the equation

$$F_n^{(k)} - 2^m = F_{n_1}^{(k)} - 2^{m_1} \quad (= c) \tag{6.3}$$

with $(n, m) \neq (n_1, m_1)$. The main result in [25] is the following.

Theorem 6.1. *Assume that $k \geq 4$. Then equation* (6.3) *with $n > n_1 \geq 2$, $m > m_1 \geq 0$ has the following families of solutions (c, n, m, n_1, m_1):*

(i) *In the range $2 \leq n_1 < n \leq k + 1$, one has the following solutions:*

$$(0, s, s-2, t, t-2) \qquad \textit{for } 2 \leq t < s \leq k+1.$$

(ii) *In the ranges* $2 \leq n_1 \leq k+1$ *and* $k+2 \leq n \leq 2k+2$, *one has the following solutions:*

(a) *when* $n_1 = n-1$,
$$\left(2^{k-1}-1, k+2, k-1, k+1, 0\right);$$

(b) *when* $n_1 < n-1$,
$$\left(2^{\gamma}-2^{\rho}, k+2^{a}-2^{b}, k+2^{a}-2^{b}-2, \gamma+2, \rho\right),$$
with $\gamma = b-3+2^{a}-2^{b}$ *and* $\rho = a-3+2^{a}-2^{b}$, *where* $a > b \geq 0$, $(a,b) \neq (1,0)$, *and* $\gamma+3 \leq k+2$.

(iii) *In the range* $k+2 \leq n_1 < n \leq 2k+2$, *one has the following solutions: if the integer* a *is maximal such that* $2^{a} \leq k+2$ *satisfies* $a+2^{a} = k+1+2^{b}$ *for some positive integer* b, *then*
$$(-2^{a+2^{a}-3}, k+2^{a}, k+2^{a}-2, k+2^{b}, b+2^{b}-3).$$

(iv) *If* $n = 2k+3$, *and additionally* $k = 2^{t}-3$ *for some integer* $t \geq 3$, *then*
$$(1-2^{t+2^{t}-3}, 2^{t+1}-3, 2^{t+1}-5, 2, t+2^{t}-3).$$

Equation (6.3) *has no solutions with* $n > 2k+3$.

7. Some related results with generalized Pell numbers

One may ask whether one can apply the same methods to study similar equations for other parametric families of sequences resembling the sequences $\mathbf{F}^{(k)}$ of k-Fibonacci numbers. In a similar vein, for an integer $k \geq 2$ the k-Pell sequence $\mathbf{P}^{(k)} := (P_n^{(k)})_{n \geq -(k-2)}$ is given by the recurrence
$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$
with the initial conditions $P_i^{(k)} = 0$ for $i = -(k-2), \ldots, 0$, and $P_1^{(k)} = 1$. We shall refer to $P_n^{(k)}$ as the n-th k-Pell number. When $k = 2$, this coincides with the classical Pell sequence. The k-Pell numbers and their properties have been recently studied in [14, 41, 42, 43].

In 2015, Faye and Luca [31] looked for repdigits in the usual Pell sequence and using some elementary methods concluded that there are no Pell numbers larger than 10 which are repdigits. In [12], Bravo and Herrera extended the previous work and searched for k-Pell numbers which are repdigits, i.e., they determined all the solutions of the Diophantine equation
$$P_n^{(k)} = a\left(\frac{10^{\ell}-1}{9}\right) \tag{7.1}$$
in positive integers n, k, ℓ, a with $k \geq 2$, $\ell \geq 2$, and $a \in \{1, 2, \ldots, 9\}$. We clarify that the condition $\ell \geq 2$ in the above equation is only meant to ensure that $P_n^{(k)}$ has at least two digits so as to avoid trivial solutions.

Before presenting the main result of [12], we mention that in the Pell case, namely when $k = 2$, several well-known divisibility properties of the Pell numbers were used by Faye and Luca in [31] to solve equation (7.1). Unfortunately, divisibility properties similar to those used in [31] are not known to hold for $\mathbf{P}^{(k)}$ when $k \geq 3$ and therefore it was necessary to attack the problem differently. Linear forms in logarithms and standard facts concerning continued fractions allowed them to prove the following result.

Theorem 7.1. *If $P_n^{(k)}$ is a repdigit with at least two digits, then $(n,k) = (5,3)$, or $(6,4)$. Namely, the only solutions are $P_5^{(3)} = 33$ and $P_6^{(4)} = 88$.*

In 2011, Alekseyev [2] established that $F^{(2)} \cap P^{(2)} = \{0,1,2,5\}$ using properties of Lucas sequences, homogeneous quadratic Diophantine equations, and Thue equations. In 2019, Bravo, Gómez, and Herrera [8] found all generalized Fibonacci numbers which are Pell numbers, while Bravo and Herrera [11] found all Fibonacci numbers which are generalized Pell numbers.

Finally, Bravo, Herrera, and Luca [13] investigated the problem of determining $\bigcup_{k\geq 2,\ell\geq 2} P^{(k)} \cap F^{(\ell)}$ extending in this way the previous results from [2, 8, 11]. We mention that the results and remarks from Section 2 apply here in the context of showing that $P^{(k)} \cap F^{(\ell)}$ is finite for fixed $k, \ell \geq 2$ since the dominant roots of $P^{(k)}$ and $F^{(\ell)}$ are multiplicatively independent for small values of k and ℓ (but see [13, Conjecture 1]). The following is the main result of [13].

Theorem 7.2. *The only solutions (n,k,m,ℓ) of the Diophantine equation*

$$P_n^{(k)} = F_m^{(\ell)}$$

in positive integers n,k,m,ℓ with $k,\ell \geq 2$ are:

(i) *the parametric family of solutions (n,k,m,ℓ) with $\ell = 2$, namely*

$$(n,k,m,\ell) = (t,k,2t-1,2) \quad \textit{for } 1 \leq t \leq k+1;$$

(ii) *the sporadic solutions*

$$\begin{aligned}
1 &= P_1^{(k)} = F_1^{(\ell)} \quad \textit{for all } k \geq 2 \textit{ and } \ell \geq 3;\\
1 &= P_1^{(k)} = F_2^{(\ell)} \quad \textit{for all } k,\ell \geq 2;\\
2 &= P_2^{(k)} = F_3^{(\ell)} \quad \textit{for all } k \geq 2 \textit{ and } \ell \geq 3;\\
13 &= P_4^{(k)} = F_6^{(3)} \quad \textit{for all } k \geq 3;\\
29 &= P_5 = F_7^{(4)}.
\end{aligned}$$

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References

[1] A. Alahmadi, A. Altassan, F. Luca, and H. Shoaib, Products of k-Fibonacci numbers which are rep-digits, *Publ. Math. Debrecen* **97** (2020), no. 1-2, 101–115. MR 4128145.

[2] M. A. Alekseyev, On the intersections of Fibonacci, Pell, and Lucas numbers, *Integers* **11** (2011), no. 3, 239–259. MR 2988061.

[3] A. Baker and G. Wüstholz, Logarithmic forms and group varieties, *J. Reine Angew. Math.* **442** (1993), 19–62. MR 1234835.

[4] M. A. Bennett, On some exponential equations of S. S. Pillai, *Canad. J. Math.* **53** (2001), no. 5, 897–922. MR 1859761.

[5] M. A. Bennett and Á. Pintér, Intersections of recurrence sequences, *Proc. Amer. Math. Soc.* **143** (2015), no. 6, 2347–2353. MR 3326017.

[6] E. F. Bravo, J. J. Bravo, and F. Luca, Coincidences in generalized Lucas sequences, *Fibonacci Quart.* **52** (2014), no. 4, 296–306. MR 3276052.

[7] E. F. Bravo, C. A. Gómez, and F. Luca, Product of consecutive tribonacci numbers with only one distinct digit, *J. Integer Seq.* **22** (2019), no. 6, Art. 19.6.3, 8 pp. MR 4020271.

[8] J. J. Bravo, C. A. Gómez, and J. L. Herrera, On the intersection of k-Fibonacci and Pell numbers, *Bull. Korean Math. Soc.* **56** (2019), no. 2, 535–547. MR 3936488.

[9] J. J. Bravo, C. A. Gómez, and F. Luca, Powers of two as sums of two k-Fibonacci numbers, *Miskolc Math. Notes* **17** (2016), no. 1, 85–100. MR 3527869.

[10] J. J. Bravo, C. A. Gómez, and F. Luca, A Diophantine equation in k-Fibonacci numbers and repdigits, *Colloq. Math.* **152** (2018), no. 2, 299–315. MR 3794331.

[11] J. J. Bravo and J. L. Herrera, Fibonacci numbers in generalized Pell sequences, *Math. Slovaca* **70** (2020), no. 5, 1057–1068. MR 4156807.

[12] J. J. Bravo and J. L. Herrera, Repdigits in generalized Pell sequences, *Arch. Math. (Brno)* **56** (2020), no. 4, 249–262. MR 4173077.

[13] J. J. Bravo, J. L. Herrera, and F. Luca, Common values of generalized Fibonacci and Pell sequences, *J. Number Theory* **226** (2021), 51–71. MR 4239716.

[14] J. J. Bravo, J. L. Herrera, and F. Luca, On a generalization of the Pell sequence, *Math. Bohem.* **146** (2021), no. 2, 199–213. MR 4261368.

[15] J. J. Bravo and F. Luca, Powers of two in generalized Fibonacci sequences, *Rev. Colombiana Mat.* **46** (2012), no. 1, 67–79. MR 2945671.

[16] J. J. Bravo and F. Luca, Coincidences in generalized Fibonacci sequences, *J. Number Theory* **133** (2013), no. 6, 2121–2137. MR 3027957.

[17] J. J. Bravo and F. Luca, On a conjecture about repdigits in k-generalized Fibonacci sequences, *Publ. Math. Debrecen* **82** (2013), no. 3-4, 623–639. MR 3066434.

[18] J. J. Bravo and F. Luca, On the largest prime factor of the k-Fibonacci numbers, *Int. J. Number Theory* **9** (2013), no. 5, 1351–1366. MR 3077719.

[19] J. J. Bravo and F. Luca, Repdigits as sums of two k-Fibonacci numbers, *Monatsh. Math.* **176** (2015), no. 1, 31–51. MR 3296202.

[20] J. J. Bravo and F. Luca, On the Diophantine equation $F_n + F_m = 2^a$, *Quaest. Math.* **39** (2016), no. 3, 391–400. MR 3510734.

[21] J. J. Bravo, F. Luca, and K. Yazán, On Pillai's problem with Tribonacci numbers and powers of 2, *Bull. Korean Math. Soc.* **54** (2017), no. 3, 1069–1080. MR 3659165.

[22] R. D. Carmichael, On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$, *Ann. of Math. (2)* **15** (1913/14), no. 1-4, 30–48 and 49–70. MR 1502458, MR 1502459.

[23] K. C. Chim, I. Pink, and V. Ziegler, On a variant of Pillai's problem II, *J. Number Theory* **183** (2018), 269–290. MR 3715237.

[24] H. Cohen, *Number Theory. Vol. I: Tools and Diophantine Equations*, Graduate Texts in Mathematics, 239, Springer, New York, 2007. MR 2312337.

[25] M. Ddamulira, C. A. Gómez, and F. Luca, On a problem of Pillai with k-generalized Fibonacci numbers and powers of 2, *Monatsh. Math.* **187** (2018), no. 4, 635–664. MR 3861322.

[26] M. Ddamulira, F. Luca, and M. Rakotomalala, On a problem of Pillai with Fibonacci numbers and powers of 2, *Proc. Indian Acad. Sci. Math. Sci.* **127** (2017), no. 3, 411–421. MR 3660342.

[27] S. Díaz Alvarado and F. Luca, Fibonacci numbers which are sums of two repdigits, in *Proceedings of the Fourteenth International Conference on Fibonacci Numbers and Their Applications*, 97–108, Aportaciones Mat. Investig., 20, Soc. Mat. Mexicana, México, 2011. MR 3243271.

[28] G. P. B. Dresden and Z. Du, A simplified Binet formula for k-generalized Fibonacci numbers, *J. Integer Seq.* **17** (2014), no. 4, Article 14.4.7, 9 pp. MR 3181762.

[29] F. Erduvan and R. Keskin, Repdigits as products of two Fibonacci or Lucas numbers, *Proc. Indian Acad. Sci. Math. Sci.* **130** (2020), no. 1, Paper No. 28, 14 pp. MR 4079185.

[30] G. Everest, A. Van der Poorten, I. Shparlinski, and T. Ward, *Recurrence Sequences*, Mathematical Surveys and Monographs, 104, American Mathematical Society, Providence, RI, 2003. MR 1990179.

[31] B. Faye and F. Luca, Pell and Pell-Lucas numbers with only one distinct digit, *Ann. Math. Inform.* **45** (2015), 55–60. MR 3438812.

[32] C. A. Gómez and F. Luca, Power of two-classes in k-generalized Fibonacci sequences, *Rev. Colombiana Mat.* **48** (2014), no. 2, 219–234. MR 3291720.

[33] C. A. Gómez and F. Luca, On the distribution of the roots of polynomial $z^k - z^{k-1} - \cdots - z - 1$, *Comment. Math. Univ. Carolin.* **62** (2021), no. 3, 291–296. MR 4331283.

[34] C. A. Gómez Ruiz and F. Luca, On the largest prime factor of the ratio of two generalized Fibonacci numbers, *J. Number Theory* **152** (2015), 182–203. MR 3319060.

[35] C. A. Gómez Ruiz and F. Luca, On prime factors of the sum of two k-Fibonacci numbers, *Int. J. Number Theory* **14** (2018), no. 4, 1171–1195. MR 3801090.

[36] C. A. Gómez Ruiz and F. Luca, On the zero-multiplicity of a fifth-order linear recurrence, *Int. J. Number Theory* **15** (2019), no. 3, 585–595. MR 3925754.

[37] A. Herschfeld, The equation $2^x - 3^y = d$ (Abstract), *Bull. Amer. Math. Soc.* **41** (1935), 631 MR 1563159.

[38] A. Herschfeld, The equation $2^x - 3^y = d$, *Bull. Amer. Math. Soc.* **42** (1936), no. 4, 231–234. MR 1563276.

[39] F. T. Howard and C. Cooper, Some identities for r-Fibonacci numbers, *Fibonacci Quart.* **49** (2011), no. 3, 231–242. MR 2833327.

[40] L. K. Hua and Y. Wang, *Applications of Number Theory to Numerical Analysis*, Springer-Verlag, Berlin-New York; Kexue Chubanshe (Science Press), Beijing, 1981. MR 0617192.

[41] E. Kiliç, The Binet formula, sums and representations of generalized Fibonacci p-numbers, *European J. Combin.* **29** (2008), no. 3, 701–711. MR 2397350.

[42] E. Kiliç, On the usual Fibonacci and generalized order-k Pell numbers, *Ars Combin.* **88** (2008), 33–45. MR 2426404.

[43] E. Kiliç and D. Taşci, The generalized Binet formula, representation and sums of the generalized order-k Pell numbers, *Taiwanese J. Math.* **10** (2006), no. 6, 1661–1670. MR 2275152.

[44] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2001. MR 1855020.

[45] F. Luca, Fibonacci and Lucas numbers with only one distinct digit, *Portugal. Math.* **57** (2000), no. 2, 243–254. MR 1759818.

[46] F. Luca, B. V. Normenyo, and A. Togbé, Repdigits as sums of four Pell numbers, *Bol. Soc. Mat. Mex. (3)* **25** (2019), no. 2, 249–266. MR 3964309.

[47] D. Marques, The proof of a conjecture concerning the intersection of k-generalized Fibonacci sequences, *Bull. Braz. Math. Soc. (N.S.)* **44** (2013), no. 3, 455–468. MR 3124745.

[48] D. Marques, On k-generalized Fibonacci numbers with only one distinct digit, *Util. Math.* **98** (2015), 23–31. MR 3410879.

[49] D. Marques and A. Togbé, On repdigits as product of consecutive Fibonacci numbers, *Rend. Istit. Mat. Univ. Trieste* **44** (2012), 393–397. MR 3019569.

[50] M. Mignotte, Intersection des images de certaines suites récurrentes linéaires, *Theoret. Comput. Sci.* **7** (1978), no. 1, 117–122. MR 0498356.

[51] M. Mignotte, Sur les conjugués des nombres de Pisot, *C. R. Acad. Sci. Paris Sér. I Math.* **298** (1984), no. 2, 21. MR 0740081.

[52] P. Mihăilescu, Primary cyclotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.* **572** (2004), 167–195. MR 2076124.

[53] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly* **67** (1960), 745–752. MR 0123521.

[54] M. D. Miller, Mathematical Notes: On Generalized Fibonacci Numbers, *Amer. Math. Monthly* **78** (1971), no. 10, 1108–1109. MR 1536552.

[55] L. Murata and C. Pomerance, On the largest prime factor of a Mersenne number, in *Number Theory*, 209–218, CRM Proc. Lecture Notes, 36, Amer. Math. Soc., Providence, RI, 2004, MR 2076597.

[56] R. Murty and S. Wong, The *ABC* conjecture and prime divisors of the Lucas and Lehmer sequences, in *Number Theory for the Millennium, III (Urbana, IL, 2000)*, 43–54, A K Peters, Natick, MA, 2002, MR 1956267.

[57] T. D. Noe and J. Vos Post, Primes in Fibonacci n-step and Lucas n-step sequences, *J. Integer Seq.* **8** (2005), no. 4, Article 05.4.4, 12 pp. MR 2165333.

[58] B. V. Normenyo, F. Luca, and A. Togbé, Repdigits as sums of three Pell numbers, *Period. Math. Hungar.* **77** (2018), no. 2, 318–328. MR 3866634.

[59] S. S. Pillai, *On* $a^x - b^y = c$, *J. Indian Math. Soc.* **2** (1936), 119–122.

[60] S. S. Pillai, A correction to the paper On $a^x - b^y = c$, *J. Indian Math. Soc.* **2** (1937), 215.

[61] I. Pink and V. Ziegler, Effective resolution of Diophantine equations of the form $u_n + u_m = wp_1^{z_1} \cdots p_s^{z_s}$, *Monatsh. Math.* **185** (2018), no. 1, 103–131. MR 3745704.

[62] A. Schinzel, On primitive prime factors of $a^n - b^n$, *Proc. Cambridge Philos. Soc.* **58** (1962), 555–562. MR 0143728.

[63] C. L. Stewart, On divisors of Lucas and Lehmer numbers, *Acta Math.* **211** (2013), no. 2, 291–314. MR 3143892.

[64] R. J. Stroeker and R. Tijdeman, Diophantine equations, in *Computational Methods in Number Theory, Part II*, 321–369, Math. Centre Tracts, 155, Math. Centrum, Amsterdam, 1982. MR 0702521.

[65] D. A. Wolfram, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* **36** (1998), no. 2, 129–145. MR 1622060.

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