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TWO-WEIGHTED ESTIMATES OF THE MULTILINEAR FRACTIONAL INTEGRAL OPERATOR BETWEEN WEIGHTED LEBESGUE AND LIPSCHITZ SPACES WITH OPTIMAL PARAMETERS

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This article is dedicated to Professor Eleonor "Pola" Harboure, beloved colleague whose vast knowledge and human kindness have always been a guidance to us

Abstract. Given an m-tuple of weights $\vec{v} = (v_1, \dots, v_m)$, we characterize the classes of pairs $(w, \vec{v})$ involved in the boundedness properties of the multilinear fractional integral operator from $\prod_{i=1}^{m} L^{p_i}\left(v_i^{p_i}\right)$ into suitable Lipschitz spaces associated to a parameter δ, $\mathcal{L}_w(\delta)$. Our results generalize some previous estimates not only for the linear case but also for the unweighted problem in the multilinear context. We emphasize the study related to the range of the parameters involved in the problem described above, which is optimal in the sense that they become trivial outside of the region obtained. We also exhibit nontrivial examples of pairs of weights in this region.

1. Introduction

In 1972 B. Muckenhoupt characterized the nonnegative functions w for which the classical Hardy–Littlewood maximal operator M is bounded in $L^p(w)$, for $1 < p < \infty$ (see [8]). More precisely, the author proved that $M : L^p(w) \hookrightarrow L^p(w)$ if and only if $w \in A_p$, that is, w satisfies the inequality

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} \leq C$$

for every cube Q. These classes became very important for many estimates in Harmonic Analysis and were further studied by many authors.

Later on, in [9], B. Muckenhoupt and R. Wheeden introduced a variant of these sets of functions, the $A_{p,q}$ classes, given by the collection of weights w such that

$$\left(\frac{1}{|Q|}\int_Q w^q\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{1/p'} \leq C,$$

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for every cube Q, where $1 < p, q < \infty$. These classes played an important role on the boundedness properties of the fractional maximal operator M_γ, $0 < \gamma < n$ and the fractional integral operator I_γ given by the expression

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy,$$

whenever the integral is finite. It was proved in [9] that if $1 < p < n/\gamma$ and $1/q = 1/p - \gamma/n$, then this operator maps $L^p(w^p)$ into $L^q(w^q)$ if and only if $w \in A_{p,q}$. For the endpoint case $p = n/\gamma$ it was also shown that the operator I_γ maps $L^{n/\gamma}(w^{n/\gamma})$ into a weighted version of the bounded mean oscillation spaces BMO if and only if $w^{-n/(n-\gamma)} \in A_1$. Although the $A_{p,q}$ classes above are a variant of A_p, they are intimately related with them. It is well-known that $w \in A_{p,q}$ is equivalent either to $w^q \in A_{1+q/p'}$ or $w^{-p'} \in A_{1+p'/q}$ (see [9]).

Later on, in [12] the author proved that for $n/\gamma \le p < n/(\gamma-1)^+$ and $\delta = \gamma - n/p$ the operator I_γ maps $L^p(w^p)$ into suitable weighted Lipschitz spaces related to the parameter δ. These spaces are a generalization of those introduced in [9] which correspond to $\delta = 0$. A two-weighted problem was also studied, giving the optimal parameters for which the associated classes of weights are nontrivial.

In [5] E. Harboure, O. Salinas and B. Viviani had introduced another class of weighted Lipschitz spaces wider than those considered in [12]. Concretely, they defined the class $\mathcal{L}_w(\delta)$ as the collection of locally integrable functions f such that

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{w^{-1}(B)|B|^{\delta/n}} \int_B |f(x) - f_B|\, dx < \infty. \tag{1.1}$$

They characterized the weights involved in the continuity properties of I_γ acting between $L^p(w)$ into $\mathcal{L}_w(\delta)$ for $1 < p < n/(\gamma-1)^+$ and $\delta = \gamma - n/p$. The class of weights turned out wider than the corresponding class considered in [12], being the same under certain additional assumptions on the weight. Inspired by that work, a two-weighted problem was also studied in [11].

Given $m \in \mathbb{N}$ and $0 < \gamma < mn$ the multilinear fractional integral operator of order m, $I_{\gamma,m}$, is defined as follows:

$$I_{\gamma,m}\vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m |x-y_i|)^{mn-\gamma}}\, d\vec{y},$$

where $\vec{f} = (f_1, f_2, \dots, f_m)$ and $\vec{y} = (y_1, y_2, \dots, y_m)$, provided the integral is finite.

The continuity properties of $I_{\gamma,m}$ were studied by several authors. For example, it was shown in [7] that if $0 < \gamma < mn$ then $I_{\gamma,m} : \prod_{i=1}^m L^{p_i} \hookrightarrow L^q$, where $1/p = \sum_{i=1}^m 1/p_i$ and $1/q = 1/p - \gamma/n$. The author also considered weighted versions of these estimates, generalizing the results of [9] to the multilinear context. On the other hand, in [1] unweighted estimates of $I_{\gamma,m}$ between $\prod_{i=1}^m L^{p_i}$ and Lipschitz-δ spaces were given, with $0 \le \delta < 1$ and $\delta = \gamma - n/p$. For other type of estimates involving multilinear versions of the fractional integral operator see also [3], [4], [6] and [13].

Recently in [2] we studied the boundedness of $I_{\gamma,m}$ between $\prod_{i=1}^m L^{p_i}(v_i^{p_i})$ into the space $\mathbb{L}_w(\delta)$ defined by the collection of locally integrable functions f such that

$$\sup_{B\subset\mathbb{R}^n} \frac{\|w\mathcal{X}_B\|_\infty}{|B|^{1+\delta/n}} \int_B |f(x)-f_B|\,dx < \infty,$$

characterizing the weights involved as those satisfying the condition $\mathbb{H}_m(\vec{p},\gamma,\delta)$ given by

$$\frac{\|w\mathcal{X}_B\|_\infty}{|B|^{(\delta-1)/n}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{(n-\gamma_i+1/m)p_i'}}\,dy\right)^{1/p_i'} \leq C. \tag{1.2}$$

The purpose of this article is to study the boundedness of the operator $I_{\gamma,m}$ between a product of weighted Lebesgue spaces into the Lipschitz space $\mathcal{L}_w(\delta)$ defined in (1.1). Our result generalizes the linear case when $p > n/\gamma$. We do not only consider adequate extensions of the one-weight estimates in the linear case proved in [5], but also a generalization of the corresponding two-weighted problem given in [11] for $m = 1$. We characterize the classes of weights for which the problem described above holds. We also show the optimal range of the parameters involved. The optimality is understood in the sense that the parameters describe certain region in which we can find concrete examples of weights belonging to the class, becoming trivial outside of it. The results obtained in this paper not only extend the results in [5] and [11] but also they generalize the unweighted multilinear results proved in [1] .

We shall now introduce the classes of weights and the notation required in order to state our main results.

Along the manuscript the multilinear parameter will be denoted by $m \in \mathbb{N}$. Let $0 < \gamma < mn$, $\delta \in \mathbb{R}$ and $\vec{p} = (p_1, p_2, \ldots, p_m)$ be an m-tuple of exponents where $1 \leq p_i \leq \infty$ for $1 \leq i \leq m$. We define p such that $1/p = \sum_{i=1}^m 1/p_i$.

We shall be dealing with a wider class of multilinear weights than those satisfying (1.2) (see [2]) and defined as follows. Given the weights w, $v_1, \ldots, v_m$, if $\vec{v} = (v_1, v_2, \ldots, v_m)$ we say that a pair $(w, \vec{v})$ belongs to the class $\mathcal{H}_m(\vec{p},\gamma,\delta)$ if there exists a positive constant C such that the inequality

$$\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{(n-\gamma_i+1/m)p_i'}}\,dy\right)^{1/p_i'} \leq C$$

holds for every ball $B = B(x_B, R)$, where $\sum_{i=1}^m \gamma_i = \gamma$, with $0 < \gamma_i < n$ for every i and x_B denotes the center of B. The integral above is understood as usual when $p_i = 1$, (see § 2 for further details).

When $m = 1$ the class given above was first introduced in [11] (for $w = v$ see also [10] for the case $\delta = 0$ and [5] for the one-weight case). In that paper the author showed nontrivial weights when $\delta \leq \min\{1, \gamma - n/p\}$. A similar restriction, as we shall prove, appears in the multilinear context.

Remark 1. It is easy to check that $\mathbb{H}_m(\vec{p},\gamma,\delta) \subset \mathcal{H}_m(\vec{p},\gamma,\delta)$ and, if $w^{-1} \in A_1$, both classes coincide. The same statement is true for the classes $\mathbb{L}_w(\delta)$ and $\mathcal{L}_w(\delta)$.

We recall that a weight w belongs to the *reverse Hölder* class RH_s, $1<s<\infty$, if there exists a positive constant C such that the inequality

$$\left(\frac{1}{|B|}\int_B w^s\right)^{1/s} \leq \frac{C}{|B|}\int_B w$$

holds for every ball B in $\mathbb{R}^n$. It is not difficult to see that $\mathrm{RH}_t \subset \mathrm{RH}_s$ whenever $1<s<t$. We also consider weights belonging to the class RH_∞, that is, the collection of weights w such that the inequality

$$\sup_B w \leq \frac{C}{|B|}\int_B w$$

holds for some positive constant C.

We are now in a position to state our main results.

Theorem 1.1. *Let $0<\gamma<mn$, $\delta\in\mathbb{R}$, and $\vec{p}$ a vector of exponents that verifies $p>n/\gamma$. Let $(w,\vec{v})$ a pair such that $v_i^{-p_i'}\in \mathrm{RH}_m$, for $i\in\mathcal{I}_2=\{1\leq i\leq m: 1<p_i\leq\infty\}$. Then the following statements are equivalent:*

(1) *The operator $I_{\gamma,m}$ is bounded from $\prod_{i=1}^m L^{p_i}(v_i^{p_i})$ to $\mathcal{L}_w(\delta)$;*
(2) *The pair $(w,\vec{v})$ belongs to $\mathcal{H}_m(\vec{p},\gamma,\delta)$.*

Observe that a reverse Hölder condition for the weights v_i is required for our theorem to hold. Although this seems to be a restriction, it does trivially hold when we consider $m=1$, as expected. A condition of this type was also required for the class $\mathbb{H}_m(\vec{p},\gamma,\delta)$ in [2].

We also notice that whilst there is no restriction on δ in the previous theorem, they arise as a consequence of the nature of the corresponding weights. The following theorem establishes the range of parameters involved in the class $\mathcal{H}_m(\vec{p},\gamma,\delta)$ where the weights are trivial, that is, $v_i=\infty$ a.e. for some i or $w=0$ a.e.

Theorem 1.2. *Let $0<\gamma<mn$, $\delta\in\mathbb{R}$, and $\vec{p}$ a vector of exponents. The following statements hold:*

(a) *If $\delta>1$ or $\delta>\gamma-n/p$ then condition $\mathcal{H}_m(\vec{p},\gamma,\delta)$ is satisfied if and only if $v_i=\infty$ on a subset of $\mathbb{R}^n$ of positive measure, for some $1\leq i\leq m$.*
(b) *The same conclusion holds if $\delta=\gamma-n/p=1$.*

In § 5 we shall exhibit nontrivial examples of pairs $(w,\vec{v})$, for which the class $\mathcal{H}_m(\vec{p},\gamma,\delta)$ is nonempty, depicting the corresponding regions described by the parameters. By Remark 1 we have that these regions include the corresponding ones given in [2].

Regarding the case when $w=\prod_{i=1}^m v_i$, which generalizes the one-weighted problem when $m=1$, we have proved in [2] that condition $\mathbb{H}_m(\vec{p},\gamma,\delta)$ reduces to the multilinear class $A_{\vec{p},\infty}$. This is the natural multilinear extension for the condition $v^{-p'}\in A_1$ on the linear setting. When $(w,\vec{v})\in\mathcal{H}_m(\vec{p},\gamma,\delta)$ and $w=\prod_{i=1}^m v_i$ we shall directly say that $\vec{v}\in\mathcal{H}_m(\vec{p},\gamma,\delta)$, that is, there exists a positive constant C

such that the inequality

$$|B|^{(1-\delta)/n}\prod_{i=1}^m\left(\int_{\mathbb{R}^n}\frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{(n-\gamma_i+1/m)p_i'}}\,dy\right)^{1/p_i'}\leq\frac{C}{|B|}\int_B\prod_{i=1}^m v_i^{-1}$$

holds for every ball B, with the obvious changes when $p_i=1$ for some i. The following theorem deals with this case of related weights.

Theorem 1.3. *Let $0<\gamma<mn$, $\delta\in\mathbb{R}$ and $\vec{p}$ a vector of exponents. If $\vec{v}\in\mathcal{H}_m(\vec{p},\gamma,\delta)$ and $p/(mp-1)>1$, then we have that $\delta=\gamma-n/p$.*

When $m=1$, the theorem above was given in [14]. As an immediate consequence we have the following result.

Corollary 1.4. *Given $0<\gamma<mn$, $\vec{p}$ a vector of exponents and $\delta=\gamma-n/p$. If $\vec{v}\in\mathcal{H}_m(\vec{p},\gamma,\delta)$ and $\alpha=\ p/(mp-1)>1$, then we have that $\prod_{i=1}^m v_i^{-1}\in\mathrm{RH}_\alpha$.*

Notice that, when $m=1$, $\alpha=p'>1$ and the corollary establishes that if $v\in\mathcal{H}_1(p,\gamma,\delta)$, then $v^{-1}\in\mathrm{RH}_{p'}$, a property proved in [5].

2. Preliminaries and definitions

Throughout the paper, C will denote an absolute constant that may change in every occurrence. By $A\lesssim B$ we mean that there exists a positive constant c such that $A\leq cB$. We say that $A\approx B$ when $A\lesssim B$ and $B\lesssim A$.

Let $m\in\mathbb{N}$. Given a set E, with E^m we shall denote the cartesian product of E m times.

It will be useful for us to consider the operator

$$J_{\gamma,m}\vec{f}(x)=\int_{(\mathbb{R}^n)^m}\left(\frac{1}{(\sum_{i=1}^m|x-y_i|)^{mn-\gamma}}-\frac{1-\mathcal{X}_{B(0,1)^m}(\vec{y})}{(\sum_{i=1}^m|y_i|)^{mn-\gamma}}\right)\prod_{i=1}^m f_i(y_i)\,d\vec{y}. \tag{2.1}$$

which differs from $I_{\gamma,m}$ only by a constant term, therefore it has the same Lipschitz norm as $I_{\gamma,m}$, so it will be enough to give the results for $J_{\gamma,m}$.

By a weight we understand any positive and locally integrable function. As we said in the introduction, given $\delta\in\mathbb{R}$ and a weight w we say that a locally integrable function $f\in\mathcal{L}_w(\delta)$ if there exists a positive constant C such that

$$\frac{1}{w^{-1}(B)|B|^{\delta/n}}\int_B|f(x)-f_B|\,dx\leq C$$

for every ball B, where $f_B=|B|^{-1}\int_B f$.

If $\delta=0$ the space $\mathcal{L}_w(\delta)$ coincides with some weighted versions of BMO spaces introduced in [10]. Concerning the unweighted case, when $0<\delta<1$ it is equivalent to the classical Lipschitz classes $\Lambda(\delta)$ given by the collection of functions f satisfying $|f(x)-f(y)|\leq C|x-y|^\delta$ and, if $-n<\delta<0$, they are Morrey spaces. On the other hand, this space was studied for example in [5] and in [11].

The class $\mathcal{H}_m(\vec{p},\gamma,\delta)$ is given by the pairs $(w,\vec{v})$ for which the inequality

$$\sup_{B\subset\mathbb{R}^n} \frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} \frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{(n-\gamma_i+1/m)p_i'}}\, dy \right)^{1/p_i'} < \infty \tag{2.2}$$

holds. For those index i such that $p_i = 1$ we understand the corresponding factor on the product above as

$$\left\| \frac{v_i^{-1}}{(|B|^{1/n}+|x_B-\cdot|)^{(n-\gamma_i+1/m)}} \right\|_\infty .$$

Let $\mathcal{I}_1 = \{1 \le i \le m : p_i = 1\}$ and $\mathcal{I}_2 = \{1 \le i \le m : p_i > 1\}$. We will also denote with m_j the cardinal of the set $\mathcal{I}_j$, that is, $m_j = \#\mathcal{I}_j$ for $j = 1, 2$. We shall use this notation throughout the paper.

Observe that if $(w,\vec{v})$ belongs to $\mathcal{H}_m(\vec{p},\gamma,\delta)$, then the inequalities

$$\frac{|B|^{1-\delta/n+\gamma/n-1/p}}{w^{-1}(B)} \prod_{i\in\mathcal{I}_1} \|v_i^{-1}\mathcal{X}_B\|_\infty \prod_{i\in\mathcal{I}_2} \left(\frac{1}{|B|}\int_B v_i^{-p_i'} \right)^{1/p_i'} \le C \tag{2.3}$$

and

$$\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \prod_{i\in\mathcal{I}_1} \left\| \frac{v_i^{-1}\mathcal{X}_{\mathbb{R}^n\setminus B}}{(|B|^{1/n}+|x_B-\cdot|)^{(n-\gamma_i+1/m)}} \right\|_\infty \times \prod_{i\in\mathcal{I}_2} \left(\int_{\mathbb{R}^n\setminus B} \frac{v_i^{-p_i'}(y)}{|x_B-y|^{(n-\gamma_i+1/m)p_i'}}\, dy \right)^{1/p_i'} \le C, \tag{2.4}$$

hold for every ball B. We shall refer to these inequalities as the *local* and the *global* conditions, respectively. Furthermore, if $\mathcal{I}$ and $\mathcal{J}$ partition the set $\mathcal{I}_1$, from (2.2) we can write

$$\frac{|B|^{1+(\gamma-\delta)/n-1/p}}{w^{-1}(B)} \prod_{i\in\mathcal{I}} \|v_i^{-1}\mathcal{X}_{2B-B}\|_\infty \prod_{i\in\mathcal{J}} \|v_i^{-1}\mathcal{X}_B\|_\infty \prod_{i\in\mathcal{I}_2} \left(\frac{1}{|2B|}\int_{2B} v_i^{-p_i'} \right)^{1/p_i'} \le C \tag{2.5}$$

for every ball B. This inequality will be useful for our purposes later.

On the other hand, when $v_i^{-1} \in \mathrm{RH}_\infty$ for $i \in \mathcal{I}_1$ and $v_i^{-p_i'}$ is doubling for $i \in \mathcal{I}_2$, the corresponding local and global conditions imply (2.2). Before stating and proving this result, we shall introduce some useful notation.

Given $m \in \mathbb{N}$ we denote $S_m = \{0,1\}^m$. Given a set B and $\sigma \in S_m$, $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$ we define

$$B^{\sigma_i} = \begin{cases} B, & \text{if } \sigma_i = 1 \\ \mathbb{R}^n\setminus B, & \text{if } \sigma_i = 0. \end{cases}$$

With the notation $\mathbf{B}^\sigma$ we will understand the cartesian product $B^{\sigma_1} \times B^{\sigma_2} \times \cdots \times B^{\sigma_m}$. In particular, if we set $\mathbf{1} = (1, 1, \dots, 1)$ and $\mathbf{0} = (0, 0, \dots, 0)$ then we have

$$\mathbf{B}^{\mathbf{1}} = B\times B\times\cdots\times B = B^m \text{ and } \mathbf{B}^{\mathbf{0}} = (\mathbb{R}^n\setminus B)\times(\mathbb{R}^n\setminus B)\times\cdots\times(\mathbb{R}^n\setminus B) = (\mathbb{R}^n\setminus B)^m.$$

Lemma 2.1. *Let $0<\gamma<mn$, $\delta\in\mathbb{R}$, $\vec{p}$ a vector of exponents and $(w,\vec{v})$ a pair of weights such that $v_i^{-1}\in \mathrm{RH}_\infty$ for $i\in\mathcal{I}_1$ and $v_i^{-p_i'}$ is doubling for $i\in\mathcal{I}_2$. Then condition $\mathcal{H}_m(\vec{p},\gamma,\delta)$ is equivalent to (2.4).*

Proof. We have already seen that $\mathcal{H}_m(\vec{p},\gamma,\delta)$ implies (2.4). In order to prove the converse, we let $\theta_i=n-\gamma_i+1/m$ for every i. Recall that $m_2=\#\mathcal{I}_2$. By splitting the integral into the ball B and its complement and by computing the product we have that

$$\prod_{i\in\mathcal{I}_2}\left(\int_{\mathbb{R}^n}\frac{v_i^{-p_i'}}{(|B|^{1/n}+|x_B-\cdot|)^{\theta_i p_i'}}\right)^{\frac{1}{p_i'}}=\sum_{\sigma\in S_{m_2}}\prod_{i=1}^{m_2}\left(\int_{B^{\sigma_i}}\frac{v_i^{-p_i'}}{(|B|^{1/n}+|x_B-\cdot|)^{\theta_i p_i'}}\right)^{\frac{1}{p_i'}},$$

where we have used a possible rearrangement and re-indexing of $i\in\mathcal{I}_2$.

Fix $\sigma\in S_{m_2}$. If $\sigma_i=0$, we have that

$$\begin{aligned}\left(\int_{B^{\sigma_i}}\frac{v_i^{-p_i'}}{(|B|^{1/n}+|x_B-\cdot|)^{\theta_i p_i'}}\right)^{\frac{1}{p_i'}}&=\left(\int_{\mathbb{R}^n\setminus B}\frac{v_i^{-p_i'}}{(|B|^{1/n}+|x_B-\cdot|)^{\theta_i p_i'}}\right)^{\frac{1}{p_i'}}\\&\leq\left(\int_{\mathbb{R}^n\setminus B}\frac{v_i^{-p_i'}(y)}{|x_B-y|^{\theta_i p_i'}}\,dy\right)^{\frac{1}{p_i'}}.\end{aligned}$$

For $\sigma_i=1$, since $v_i^{-p_i'}$ is doubling, we have that

$$\begin{aligned}\left(\int_{B^{\sigma_i}}\frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{\theta_i p_i'}}\,dy\right)^{\frac{1}{p_i'}}&=\left(\int_{B}\frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{\theta_i p_i'}}\,dy\right)^{\frac{1}{p_i'}}\\&\leq\frac{1}{|B|^{1-\gamma_i/n+1/(mn)}}\left(\int_B v_i^{-p_i'}\right)^{1/p_i'}\\&\lesssim\frac{1}{|2B|^{1-\gamma_i/n+1/(mn)}}\left(\int_{2B\setminus B} v_i^{-p_i'}\right)^{1/p_i'}\\&\leq\left(\int_{2B\setminus B}\frac{v_i^{-p_i'}(y)}{|x_B-y|^{(n-\gamma_i+1/m)p_i'}}\right)^{1/p_i'}\\&\leq\left(\int_{\mathbb{R}^n\setminus B}\frac{v_i^{-p_i'}(y)}{|x_B-y|^{(n-\gamma_i+1/m)p_i'}}\right)^{1/p_i'}.\end{aligned}$$

Therefore, for every $\sigma\in S_{m_2}$, we obtain

$$\prod_{i=1}^{m_2}\left(\int_{B^{\sigma_i}}\frac{v_i^{-p_i'}}{(|B|^{1/n}+|x_B-\cdot|)^{\theta_i p_i'}}\right)^{1/p_i'}\lesssim\prod_{i\in\mathcal{I}_2}\left(\int_{\mathbb{R}^n\setminus B}\frac{v_i^{-p_i'}}{|x_B-\cdot|^{\theta_i p_i'}}\right)^{1/p_i'}. \tag{2.6}$$

On the other hand, for $i \in \mathcal{I}_1$ we proceed similarly as above replacing $\|\cdot\|_{p_i'}$ by $\|\cdot\|_\infty$ and using the RH_∞ condition for v_i^{-1}. Indeed, observe that

$$\left\|v^{-1}\mathcal{X}_B\right\|_\infty \leq \frac{C}{|B|}\int_B v^{-1} \leq \frac{C}{|B|}\int_{2B\setminus B} v^{-1} \leq C\left\|v^{-1}\mathcal{X}_{2B\setminus B}\right\|_\infty .$$

Then we can conclude that

$$\prod_{i\in\mathcal{I}_1}\left\|\frac{v_i^{-1}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\gamma/m+1/m}}\right\|_\infty \lesssim \prod_{i\in\mathcal{I}_1}\left\|\frac{v_i^{-1}\mathcal{X}_{\mathbb{R}^n\setminus B}}{|x_B-\cdot|^{n-\gamma/m+1/m}}\right\|_\infty . \tag{2.7}$$

Therefore, by combining (2.6), (2.7) and (2.4) we get that

$$\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)}\prod_{i\in\mathcal{I}_1}\left\|\frac{v_i^{-1}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\gamma/m+1/m}}\right\|_\infty$$
$$\times\prod_{i\in\mathcal{I}_2}\left(\int_{\mathbb{R}^n}\frac{v_i^{-p_i'}}{(|B|^{1/n}+|x_B-\cdot|)^{\theta_i p_i'}}\right)^{1/p_i'} \leq C,$$

as desired. □

Corollary 2.2. *Under the hypotheses of Lemma 2.1 we have that condition* (2.4) *implies* (2.3).

3. Technical results

We now introduce some operators related to $I_{\gamma,m}$ and some useful properties in order to prove our main results.

Given a ball $B = B(x_B, R)$ and $\tilde{B} = 2B$, as in [2] we can decompose the operator in (2.1) as

$$J_{\gamma,m}\vec{f}(x) = a_B + I\vec{f}(x),$$

where

$$a_B = \int_{(\mathbb{R}^n)^m}\left(\frac{1-\mathcal{X}_{\tilde{B}^m}(\vec{y})}{(\sum_{i=1}^m|x_B-y_i|)^{mn-\gamma}} - \frac{1-\mathcal{X}_{B(0,1)^m}(\vec{y})}{(\sum_{i=1}^m|y_i|)^{mn-\gamma}}\right)\prod_{i=1}^m f_i(y_i)\,d\vec{y} \tag{3.1}$$

and

$$I\vec{f}(x) = \int_{(\mathbb{R}^n)^m}\left(\frac{1}{(\sum_{i=1}^m|x-y_i|)^{mn-\gamma}} - \frac{1-\mathcal{X}_{\tilde{B}^m}(\vec{y})}{(\sum_{i=1}^m|x_B-y_i|)^{mn-\gamma}}\right)\prod_{i=1}^m f_i(y_i)\,d\vec{y}. \tag{3.2}$$

We shall first prove that this operator is well-defined for $\vec{f}$ as in Theorem 1.1.

Lemma 3.1. *Let $0<\gamma<mn$, $\delta\in\mathbb{R}$, and $\vec{p}$ a vector of exponents that verifies $p>n/\gamma$. Let $(w,\vec{v})$ be a pair of weights in $\mathcal{H}_m(\vec{p},\gamma,\delta)$ such that $v_i^{-p_i'}\in\mathrm{RH}_m$, for $i\in\mathcal{I}_2$. If $\vec{f}$ satisfies $f_iv_i\in L^{p_i}$ for every $1\leq i\leq m$, then $J_{\gamma,m}\vec{f}$ is finite in almost every $x\in\mathbb{R}^n$.*

Proof. We are going to exhibit a sketch of the proof, since it follows similar lines to that in [2, Lemma 3.1]. By using the same notation as in that lemma, fix a ball $B = B(x_B, R)$ and write $J_{\gamma,m}\vec{f} = a_B + I\vec{f}$, where we split $a_B = a_B^1 + a_B^2$ and $I\vec{f} = I_1\vec{f} + I_2\vec{f}$. We proved that

$$|a_B^1| \le \left(1 + \frac{C}{|B|^{m-\gamma/n}}\right) \prod_{i=1}^m \left(\int_{B_0} |f_i(y_i)|\, d_{y_i}\right),$$

where $B_0 = B(0, R_0)$ with $R_0 = 2(|x_B| + R)$. By using Hölder's inequality and condition (2.3) we get

$$\begin{aligned} |a_B^1| &\le \left(1 + \frac{C}{|B|^{m-\gamma/n}}\right) \prod_{i=1}^m \|f_i v_i\|_{p_i} \prod_{i\in\mathcal{I}_1} \left\|v_i^{-1}\mathcal{X}_{B_0}\right\|_\infty \prod_{i\in\mathcal{I}_2} \left(\int_{B_0} v_i^{-p_i'}\right)^{1/p_i'} \\ &\le \left(1 + \frac{C}{|B|^{m-\gamma/n}}\right) \prod_{i=1}^m \|f_i v_i\|_{p_i} \frac{w^{-1}(B_0)}{|B_0|} |B_0|^{\delta/n-\gamma/n+1/p} \\ &< \infty. \end{aligned}$$

In the same lemma we also proved that

$$\begin{aligned} |a_B^2| \le C \prod_{i=1}^m \|f_i v_i\|_{p_i} \prod_{i\in\mathcal{I}_1} \left\| \frac{v_i^{-1}}{(|B_0|^{1/n} + |x_{B_0} - \cdot|)^{\theta_i}} \right\|_\infty \\ \times \prod_{i\in\mathcal{I}_2} \left(\int_{\mathbb{R}^n} \frac{v_i^{-p_i'}}{(|B_0|^{1/n} + |x_{B_0} - y_i|)^{\theta_i p_i'}} \right)^{1/p_i'}, \end{aligned}$$

where $\theta_i = n - \gamma_i + 1/m$. So by using condition (2.2) we get that

$$|a_B^2| \le C\frac{w^{-1}(B_0)}{|B_0|} |B_0|^{(\delta-1)/n} \prod_{i=1}^m \|f_i v_i\|_{p_i} < \infty.$$

Let us now consider $I_1\vec{f}$. By proceeding as in the corresponding estimate in [2] we obtain

$$\begin{aligned} \int_B |I_1\vec{f}(x)|\, dx &\le C \prod_{i=1}^m \|f_i v_i\|_{p_i} \prod_{i\in\mathcal{I}_2} \left(\frac{1}{|\tilde{B}|}\int_{\tilde{B}} v_i^{-p_i'}\right)^{1/p_i'} \\ &\quad \times \prod_{i\in\mathcal{I}_1} \left\|v_i^{-1}\mathcal{X}_{\tilde{B}}\right\|_\infty |\tilde{B}|^{(\gamma-\gamma_0)/n-m_1+1/q'+1-1/(m_0 p^*)} \\ &= C|\tilde{B}|^{\gamma/n-1/p+1} \prod_{i=1}^m \|f_i v_i\|_{p_i} \prod_{i\in\mathcal{I}_1} \left\|v_i^{-1}\mathcal{X}_{\tilde{B}}\right\|_\infty \prod_{i\in\mathcal{I}_2} \left(\frac{1}{|\tilde{B}|}\int_{\tilde{B}} v_i^{-p_i'}\right)^{1/p_i'}. \end{aligned}$$

We rearrange the indices in $\mathcal{I}_1$ increasingly, in a way to get $\mathcal{I}_1=\{i_1,\dots,i_{m_1}\}$. Observe that

$$\prod_{i\in\mathcal{I}_1}\left\|v_i^{-1}\mathcal{X}_{\tilde B}\right\|_\infty\le\prod_{i\in\mathcal{I}_1}\left(\left\|v_i^{-1}\mathcal{X}_{\tilde B-B}\right\|_\infty+\left\|v_i^{-1}\mathcal{X}_B\right\|_\infty\right)$$
$$=\sum_{\sigma\in S^{m_1}}\prod_{j=1}^{m_1}\left\|v_{i_j}^{-1}\mathcal{X}_{\tilde B-B}\right\|_\infty^{\sigma_j}\left\|v_{i_j}^{-1}\mathcal{X}_B\right\|_\infty^{1-\sigma_j}.$$

Therefore,

$$\int_B|I_1\vec f(x)|\,dx\le C\prod_{i=1}^m\|f_iv_i\|_{p_i}\sum_{\sigma\in S^{m_1}}|\tilde B|^{\gamma/n-1/p+1}\prod_{i\in\mathcal{I}_2}\left(\frac{1}{|\tilde B|}\int_{\tilde B}v_i^{-p_i'}\right)^{1/p_i'}$$
$$\times\prod_{j=1}^{m_1}\left\|v_{i_j}^{-1}\mathcal{X}_{\tilde B-B}\right\|_\infty^{\sigma_j}\left\|v_{i_j}^{-1}\mathcal{X}_B\right\|_\infty^{1-\sigma_j}.$$

Fix $\sigma\in S^{m_1}$ and define the sets

$$\mathcal{I}=\{i_j\in\mathcal{I}_1:\sigma_j=1\}\quad\text{ and }\quad\mathcal{J}=\{i_j\in\mathcal{I}_1:\sigma_j=0\}.$$

We can apply condition (2.5) to bound every term of the sum by

$$C\frac{w^{-1}(B)}{|B|^{1+(\gamma-\delta)/n-1/p}}|\tilde B|^{\gamma/n-1/p+1}=Cw^{-1}(B)|B|^{\delta/n}.$$

Consequently,

$$\int_B|I_1\vec f(x)|\,dx\le Cw^{-1}(B)|B|^{\delta/n}\left(\prod_{i=1}^m\|f_iv_i\|_{p_i}\right).$$

Finally, for $I_2\vec f$ we have

$$|I_2\vec f(x)|\le|B|^{1/n}\sum_{\sigma\in S_m,\sigma\neq\mathbf{1}}\int_{\tilde{\mathbf{B}}^\sigma}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x_B-y_i|)^{mn-\gamma+1}}\,d\vec y.$$

This expression is similar to a_B^2, with B_0 replaced by $\tilde B$. Observe that

$$\left\|\frac{v_i^{-1}}{|x_B-\cdot|^{\theta_i}}\mathcal{X}_{\tilde B^c}\right\|_\infty\le\left\|\frac{v_i^{-1}}{|x_B-\cdot|^{\theta_i}}\mathcal{X}_{B^c}\right\|_\infty$$

for those indices $i\in\mathcal{I}_1$ such that $\sigma_i=0$. On the other hand, if $i\in\mathcal{I}_1$ and $\sigma_i=1$, we can split the expression $\|v_i^{-1}\mathcal{X}_{\tilde B}\|_\infty$ as follows:

$$\left\|v_i^{-1}\mathcal{X}_{\tilde B}\right\|_\infty\le\left\|v_i^{-1}\mathcal{X}_{\tilde B-B}\right\|_\infty+\left\|v_i^{-1}\mathcal{X}_B\right\|_\infty$$

and repeat the argument used in the estimation of $I_1\vec f$. After applying condition (2.5) we get that

$$\int_B|I_2\vec f(x)|\,dx\le Cw^{-1}(B)|B|^{\delta/n}\prod_{i=1}^m\|f_iv_i\|_{p_i}.$$

This concludes the proof of the lemma. □

Remark 2. The corresponding bound obtained for $I\vec{f}$ will be used for the proof of Theorem 1.1.

The next lemma was given in [2]. The sets involved in its statement are defined as follows.

For a fixed ball $B = B(x_B, R)$ we set

$$A = \{x_B + h : h = (h_1, h_2, \dots, h_n) : h_i \geq 0 \text{ for } 1 \leq i \leq n\},$$

$$C_1 = B\left(x_B - \frac{R}{12\sqrt{n}}u, \frac{R}{12\sqrt{n}}\right) \cap \left\{x_B - \frac{R}{12\sqrt{n}}u + h : h_i \leq 0 \text{ for every } i\right\},$$

and

$$C_2 = B\left(x_B - \frac{R}{3\sqrt{n}}u, \frac{2R}{3}\right) \cap \left\{x_B - \frac{R}{3\sqrt{n}}u + h : h_i \leq 0 \text{ for every } i\right\},$$

where $u = (1, 1, \dots, 1)$.

Lemma 3.2. *There exists a positive constant $C = C(n)$ such that the inequality*

$$\frac{1}{(\sum_{j=1}^m |x - y_j|)^{mn-\gamma}} - \frac{1}{(\sum_{j=1}^m |z - y_j|)^{mn-\gamma}} \geq C \frac{|B|^{1/n}}{(|B|^{1/n} + \sum_{j=1}^m |x_B - y_j|)^{mn-\gamma+1}}$$

holds for every $x \in C_1$, $z \in C_2$, and $y_j \in A$ for $1 \leq j \leq m$.

Remark 3. It is not difficult to see that $|C_i| \approx |B|$ for $i = 1, 2$.

4. Proof of the main results

In this section we prove our main results.

Proof of Theorem 1.1. We shall first prove that (2) implies (1). We shall deal with the operator $J_{\gamma,m}$ since it differs from $I_{\gamma,m}$ by a constant term. We want to prove that for every ball B

$$\frac{1}{w^{-1}(B)|B|^{\delta/n}} \int_B |J_{\gamma,m}\vec{f}(x) - (J_{\gamma,m}\vec{f})_B|\, dx \leq C \prod_{i=1}^m \|f_i v_i\|_{p_i}, \tag{4.1}$$

with C independent of B. Fix a ball $B = B(x_B, R)$ and recall that $J_{\gamma,m}\vec{f}(x) = a_B + I\vec{f}(x)$, where a_B is given by (3.1) and $I\vec{f}$ by (3.2). In Lemma 3.1 we proved that

$$\int_B |I\vec{f}(x)|\, dx \leq C w^{-1}(B)|B|^{\delta/n} \prod_{i=1}^m \|f_i v_i\|_{p_i},$$

which implies that

$$\int_B |J_{\gamma,m}\vec{f}(x) - a_B|\, dx \leq C w^{-1}(B)|B|^{\delta/n} \prod_{i=1}^m \|f_i v_i\|_{p_i}. \tag{4.2}$$

On the other hand, observe that

$$\begin{aligned}\int_B |J_{\gamma,m}\vec{f}(x)-(J_{\gamma,m}\vec{f}\,)_B|\,dx &\le \int_B |J_{\gamma,m}\vec{f}(x)-a_B|\,dx+\int_B |(J_{\gamma,m}\vec{f}\,)_B-a_B|\,dx\\ &\le 2\int_B |J_{\gamma,m}\vec{f}(x)-a_B|\,dx.\end{aligned}$$

By combining this estimate with (4.2) we obtain the desired inequality.

We now prove that (1) implies (2). Assume that the component functions f_i of $\vec{f}$ are nonnegative. We have that (4.1) holds for every ball $B=B(x_B,R)$. Also observe that

$$\frac{1}{|B|}\int_B |g(x)-g_B|\,dx \approx \frac{1}{|B|^2}\int_B\int_B |g(x)-g(z)|\,dx\,dz,$$

and therefore the left-hand side of (4.1) is equivalent to

$$\frac{1}{w^{-1}(B)|B|^{1+\delta/n}}\int_B\int_B |J_{\gamma,m}\vec{f}(x)-J_{\gamma,m}\vec{f}(z)|\,dx\,dz=I.$$

Observe that, when $y_i\in B$ for every i we have

$$|B|^{1/n}+|x_B-y_j|\ge \frac{1}{m}\left(|B|^{1/n}+\sum_{i=1}^m |x_B-y_i|\right),$$

for every $1\le j\le m$. By combining Lemma 3.2 and Remark 3 with the inequality above we can estimate I as follows

$$\begin{aligned}I &\ge \frac{1}{w^{-1}(B)|B|^{1+\delta/n}}\int_{C_2}\int_{C_1}\int_{A^m}\frac{|B|^{1/n}\prod_{i=1}^m f_i(y_i)}{(|B|^{1/n}+\sum_{i=1}^m|x_B-y_i|)^{mn-\gamma+1}}\,d\vec{y}\,dx\,dz\\ &\ge C\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)}\prod_{i=1}^m\left(\int_A \frac{f_i(y_i)}{(|B|^{1/n}+|x_B-y_i|)^{n-\gamma_i+1/m}}\,dy_i\right).\end{aligned}$$

Since the set A is a quadrant from x_B, a similar estimation can be obtained for the other quadrants from x_B. Thus, we get

$$I\ge C\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)}\prod_{i=1}^m\left(\int_{\mathbb{R}^n}\frac{f_i(y)}{(|B|^{1/n}+|x_B-y|)^{n-\gamma_i+1/m}}\,dy\right),$$

which implies that

$$\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)}\prod_{i=1}^m\left(\int_{\mathbb{R}^n}\frac{f_i(y)}{(|B|^{1/n}+|x_B-y|)^{n-\gamma_i+1/m}}\,dy\right)\le C\prod_{i=1}^m\|f_iv_i\|_{p_i}. \tag{4.3}$$

For every $i\in\mathcal{I}_1$ and $k\in\mathbb{N}$ we define $V_k^i=\{x: v_i^{-1}(x)\le k\}$ and the functionals

$$F_i^k(g)=\int_{\mathbb{R}^n}\frac{g(y)v_i^{-1}(y)\mathcal{X}_{V_k^i}(y)}{(|B|^{1/n}+|x_B-y|)^{n-\gamma_i+1/m}}\,dy.$$

Therefore F_i^k is a functional in $(L^1)^*=L^\infty$. Indeed, if $g\in L^1$,

$$|F_i^k(g)|\le \|g\|_{L^1}\left\|\frac{v_i^{-1}\mathcal{X}_{V_k^i}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\gamma_i+1/m}}\right\|_\infty<\infty,$$

and we also get

$$\frac{|F_i^k(f_iv_i)|}{\|f_iv_i\|_{L^1}} \leq \left\| \frac{v_i^{-1}\mathcal{X}_{V_k^i}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\gamma_i+1/m}} \right\|_\infty$$

for every $i \in \mathcal{I}_1$.

If $i \in \mathcal{I}_2$, then we set $A_k = A \cap B(0,k)$ and consider

$$f_i^k(y) = \frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{(n-\gamma_i+1/m)/(p_i-1)}}\mathcal{X}_{A_k}(y)\mathcal{X}_{V_k^i}(y).$$

Let us choose $\vec{f} = (f_1, \dots, f_m)$, where $f_iv_i \in L^1$ for $p_i = 1$ and $f_i = f_i^k$ for $p_i > 1$, for fixed k. Therefore, the left-hand side of (4.3) can be written as follows:

$$\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \prod_{i\in\mathcal{I}_1} F_i^k(f_iv_i) \prod_{i\in\mathcal{I}_2} \left(\int_{A_k\cap V_k^i} \frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{(n-\gamma_i+1/m)p_i'}}\,dy \right)$$

and it is bounded by

$$C \prod_{i\in\mathcal{I}_1} \|f_iv_i\|_{L^1} \prod_{i\in\mathcal{I}_2} \left(\int_{A_k\cap V_k^i} \frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{(n-\gamma_i+1/m)p_i'}}\,dy \right)^{1/p_i}.$$

This yields

$$\begin{aligned}\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \prod_{i\in\mathcal{I}_1} \frac{|F_i^k(f_iv_i)|}{\|f_iv_i\|_{L^1}} \prod_{i\in\mathcal{I}_2} \left(\int_{A_k\cap V_k^i} \frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{(n-\gamma_i+1/m)p_i'}}\,dy \right)^{1/p_i'} \\ \leq C,\end{aligned}$$

for every nonnegative f_i such that $f_iv_i \in L^1$, $i \in \mathcal{I}_1$ and for every $k \in \mathbb{N}$. By taking the supremum over f_i iteratively for $i \in \mathcal{I}_1$ we get

$$\begin{aligned}\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \prod_{i\in\mathcal{I}_1} \left\| \frac{v_i^{-1}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\gamma_i+1/m}} \right\|_\infty \\ \times \prod_{i\in\mathcal{I}_2} \left(\int \frac{v_i^{-p_i'}\mathcal{X}_{A_k\cap V_k^i}}{(|B|^{1/n}+|x_B-\cdot|)^{(n-\gamma_i+1/m)p_i'}} \right)^{\frac{1}{p_i'}} \leq C.\end{aligned}$$

By taking limit for $k \to \infty$, the left-hand side converges to

$$\begin{aligned}\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \prod_{i\in\mathcal{I}_1} \left\| \frac{v_i^{-1}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\gamma_i+1/m}} \right\|_\infty \\ \times \prod_{i\in\mathcal{I}_2} \left(\int_{\mathbb{R}^n} \frac{v_i^{-p_i'}(y)}{(|B|^{1/n}+|x_B-y|)^{(n-\gamma_i+1/m)p_i'}}\,dy \right)^{\frac{1}{p_i'}}\end{aligned}$$

which is precisely the condition $\mathcal{H}_m(\vec{p}, \gamma, \delta)$. This completes the proof. □

Proof of Theorem 1.2. We begin with item (a). We shall first assume that $\delta > 1$. If $(w, \vec{v}) \in \mathcal{H}_m(\vec{p}, \gamma, \delta)$, we choose $B = B(x_B, R)$, where x_B is a Lebesgue point of w^{-1}. From (2.2) we obtain

$$\prod_{i\in\mathcal{I}_1} \left\| \frac{v_i^{-1}}{(|B|^{1/n} + |x_B - \cdot|)^{n-\gamma_i+1/m}} \right\|_\infty \prod_{i\in\mathcal{I}_2} \left(\int_{\mathbb{R}^n} \frac{v_i^{-p_i'}}{(|B|^{1/n} + |x_B - \cdot|)^{(n-\gamma_i+1/m)p_i'}} \right)^{\frac{1}{p_i'}} \lesssim \frac{w^{-1}(B)}{|B|R^{1-\delta}},$$

for every $R > 0$. By letting $R \to 0$ and applying the monotone convergence theorem, we conclude that at least one limit factor in the product should be zero. That is, there exists $1 \le i \le m$ such that $v_i = \infty$ almost everywhere.

On the other hand, if $\delta > \gamma - n/p$ and $(w, \vec{v})$ belongs to $\mathcal{H}_m(\vec{p}, \gamma, \delta)$, we pick a ball $B = B(x_B, R)$, where x_B is a Lebesgue point of w^{-1} and every v_i^{-1}. Then, by applying (2.3), we have

$$\begin{aligned} \prod_{i=1}^m \frac{1}{|B|} \int_B v_i^{-1} &\le \prod_{i\in\mathcal{I}_1} \left\| v_i^{-1}\mathcal{X}_B \right\|_\infty \prod_{i\in\mathcal{I}_2} \left(\frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} \\ &\le C\frac{w^{-1}(B)}{|B|} R^{\delta-\gamma+n/p} \end{aligned}$$

for every $R > 0$. By letting $R \to 0$ we get

$$\prod_{i=1}^m v_i^{-1}(x_B) = 0,$$

which yields that $\prod_{i=1}^m v_i^{-1}$ is zero almost everywhere. This implies that $M = \bigcap_{i=1}^m \{v_i^{-1} > 0\}$ has null measure. Then there exists j and a set $E_j \subset \mathbb{R}^n$ of positive measure such that $v_j = \infty$ in almost every point of E_j.

We turn now our attention to item (b), that is, $\delta = \gamma - n/p = 1$. We shall prove that if $(w, \vec{v}) \in \mathcal{H}_m(\vec{p}, \gamma, 1)$, there exists j such that $v_j = \infty$ almost everywhere in $\mathbb{R}^n$. We define

$$\frac{1}{\alpha} = \sum_{i=1}^m \frac{1}{p_i'} = \frac{mp-1}{p}.$$

By applying Hölder's inequality we obtain

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \frac{(\prod_{i\in\mathcal{I}_2} v_i^{-1})^\alpha}{(|B|^{1/n} + |x_B - y|)^{\sum_{i\in\mathcal{I}_2}(n-\gamma_i+1/m)\alpha}} \right)^{1/\alpha} \\ &\qquad \le C \prod_{i\in\mathcal{I}_2} \left(\int_{\mathbb{R}^n} \frac{v_i^{-p_i'}}{(|B|^{1/n} + |x_B - \cdot|)^{(n-\gamma_i+1/m)p_i'}} \right)^{\frac{1}{p_i'}} \end{aligned}$$

and since $(w,\vec{v})\in\mathcal{H}_m(\vec{p},\gamma,1)$, this implies that

$$\prod_{i\in\mathcal{I}_1}\left\|\frac{v_i^{-1}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\gamma_i+1/m}}\right\|_\infty\left(\int_{\mathbb{R}^n}\frac{(\prod_{i\in\mathcal{I}_2}v_i^{-1})^\alpha}{(|B|^{1/n}+|x_B-y|)^{\sum_{i\in\mathcal{I}_2}(n-\gamma_i+1/m)\alpha}}\right)^{\frac{1}{\alpha}},$$
$$\lesssim\frac{w^{-1}(B)}{|B|}$$

and furthermore

$$\left(\int_{\mathbb{R}^n}\frac{(\prod_{i=1}^m v_i^{-1})^\alpha}{(|B|^{1/n}+|x_B-y|)^{(mn-\gamma+1)\alpha}}\right)^{1/\alpha}\lesssim\frac{w^{-1}(B)}{|B|}$$

for every ball $B=B(x_B,R)$.

If we assume that the set $E=\{x:\ \prod_{i=1}^m v_i^{-1}(x)>\ 0\}$ has positive measure, we arrive at a contradiction by following the same argument as in item (b) from [2, Theorem 1.2]. This yields $|E|=0$, that is, $\prod_{i=1}^m v_i^{-1}=0$ almost everywhere, from where we can deduce that there exists an index j satisfying $v_j=\infty$ almost everywhere. □

5. The class $\mathcal{H}_m(\vec{p},\gamma,\delta)$

We begin this section by exhibiting nontrivial pairs of weights satisfying condition $\mathcal{H}_m(\vec{p},\gamma,\delta)$. Concretely, we shall prove the following theorem.

Theorem 5.1. *Given $0<\gamma<mn$ there exist pairs of weights $(w,\vec{v})$ satisfying* (2.2) *for every $\vec{p}$ and δ such that $\delta\le\min\{1,\gamma-n/p\}$, excluding the case $\delta=1$ when $\gamma-n/p=1$.*

Figure 1 shows the area in which we can find nontrivial weights satisfying condition $\mathcal{H}_m(\vec{p},\gamma,\delta)$, split into the cases $\gamma<1,\gamma=1$ and $\gamma>1$.

The following lemma will be useful in order to prove Theorem 5.1 (see [12]).

Lemma 5.2. *If $R>0$, $B=B(x_B,R)$ is a ball in $\mathbb{R}^n$ and $\alpha>-n$, then*

$$\int_B|x|^\alpha\,dx\approx R^n\left(\max\{R,|x_B|\}\right)^\alpha.$$

Proof of Theorem 5.1. In [2] we exhibited examples of weights in the class $\mathbb{H}_m(\vec{p},\gamma,\delta)$ given by (1.2), for $\gamma-mn\le\delta\le\min\{1,\gamma-n/p\}$, excluding the case $\delta=1$ when $\gamma-n/p=1$. By Remark 1 the same examples satisfy $\mathcal{H}_m(\vec{p},\gamma,\delta)$, so it will be enough to check the case $\delta<\gamma-mn$.

Recall that $\theta_i=n/p_i+(1-\gamma)/m$ and $\mathcal{I}_1=\{1\le i\le m: p_i=1\}$. Let us first assume that $\mathcal{I}_1\neq\emptyset$. Since $\gamma<mn$, we can choose $-\theta_i<\beta_i<n/p_i'$ for every $i\in\mathcal{I}_2$ and $\theta_i<0$, and $0<\beta_i<n/p_i'$ if $\theta_i\ge0$. This election implies that

$$\nu=\sum_{i\in\mathcal{I}_2,\theta_i\ge0}\beta_i+\sum_{i\in\mathcal{I}_2,\theta_i<0}(\beta_i+\theta_i)>0.$$

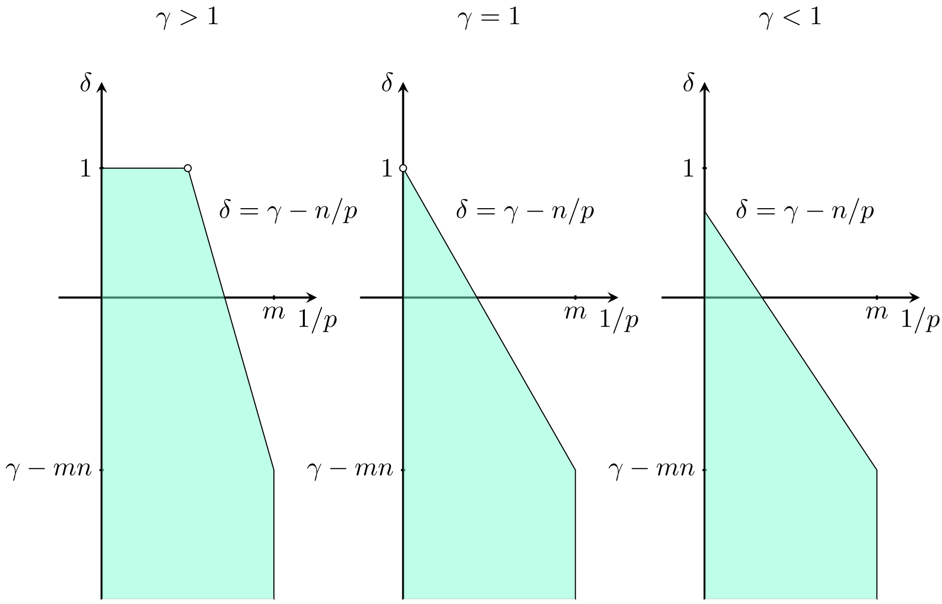

Figure 1.

We now choose

$$0 < \beta < \min\left\{\frac{\nu}{m_1}, n + \frac{1-\gamma}{m}\right\},$$

and take $\beta_i = -\beta$ for every $i \in \mathcal{I}_1$. Let $\alpha = \delta + \sum_{i=1}^m \beta_i + n/p - \gamma$ and define

$$w(x) = |x|^\alpha \quad \text{and} \quad v_i(x) = |x|^{\beta_i}, \quad \text{for } 1 \le i \le m.$$

Notice that

$$\alpha = \delta + \sum_{i=1}^m \beta_i + n/p - \gamma < \delta + \sum_{i=1}^m \frac{n}{p_i'} + \frac{n}{p} - \gamma = \delta + mn - \gamma < 0,$$

since $\delta < \gamma - mn$, so w^{-1} is a locally integrable function. On the other hand, since $v_i^{-1} \in \mathrm{RH}_\infty$ for $i \in \mathcal{I}_1$ the same conclusion holds for these weights. For $i \in \mathcal{I}_2$ we also have that $v_i^{-p_i'}$ is locally integrable since $\beta_i < n/p_i'$. Therefore, by virtue of Lemma 2.1, it will be enough to show that there exists a positive constant C such that the inequality

$$\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \prod_{i\in\mathcal{I}_1} \left\| \frac{v_i^{-1}\mathcal{X}_{\mathbb{R}^n\setminus B}}{|x_B - \cdot|^{n-\gamma/m+1/m}} \right\|_\infty \prod_{i\in\mathcal{I}_2} \left(\int_{\mathbb{R}^n\setminus B} \frac{v_i^{-p_i'}}{|x_B - \cdot|^{(n-\gamma/m+1/m)p_i'}} \right)^{1/p_i'} \le C \tag{5.1}$$

holds for every ball $B = B(x_B, R)$. We shall first assume that $|x_B| \leq R$. By Lemma 5.2 we have that

$$\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \lesssim R^{1-\delta+\alpha}. \tag{5.2}$$

On the other hand, if $i \in \mathcal{I}_1$ and $B_k = B(x_B, 2^k R)$, $k \in \mathbb{N}$, we have

$$\begin{aligned}\left\| \frac{v_i^{-1}\mathcal{X}_{\mathbb{R}^n \setminus B}}{|x_B - \cdot|^{n-\gamma/m+1/m}} \right\|_\infty &\lesssim \sum_{k=0}^{\infty} \left\| \frac{v_i^{-1}\mathcal{X}_{B_{k+1}\setminus B_k}}{|x_B - \cdot|^{n-\gamma/m+1/m}} \right\|_\infty \\ &\lesssim \sum_{k=0}^{\infty} \left(2^k R\right)^{-\beta_i - n + \gamma/m - 1/m} \\ &\lesssim R^{-\beta_i - n + \gamma/m - 1/m},\end{aligned}$$

since $-\beta_i - n + \gamma/m - 1/m < 0$. This yields

$$\prod_{i \in \mathcal{I}_1} \left\| \frac{v_i^{-1}\mathcal{X}_{\mathbb{R}^n \setminus B}}{|x_B - \cdot|^{n-\gamma/m+1/m}} \right\|_\infty \lesssim R^{-\sum_{i\in\mathcal{I}_1}(\beta_i+\theta_i)}. \tag{5.3}$$

Finally, since $\beta_i + \theta_i > 0$ for $i \in \mathcal{I}_2$, by Lemma 5.2 we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^n\setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n+\frac{1-\gamma}{m})p_i'}}\, dy\right)^{1/p_i'} &\lesssim \sum_{k=0}^{\infty} (2^k R)^{-n+\frac{\gamma-1}{m}} \left(\int_{B_{k+1}\setminus B_k} |y|^{-\beta_i p_i'}\, dy\right)^{1/p_i'} \\ &\lesssim \sum_{k=0}^{\infty} (2^k R)^{-n+\gamma/m-1/m-\beta_i+n/p_i'} \\ &\lesssim R^{-n/p_i+\gamma/m-1/m-\beta_i}.\end{aligned}$$

Therefore, we obtain

$$\prod_{i\in\mathcal{I}_2} \left(\int_{\mathbb{R}^n\setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n-\gamma/m+1/m)p_i'}}\, dy\right)^{1/p_i'} \lesssim R^{-\sum_{i\in\mathcal{I}_2}(\beta_i+\theta_i)}. \tag{5.4}$$

By combining (5.2), (5.3) and (5.4), the left-hand side of (5.1) is bounded by

$$CR^{1-\delta+\alpha-\sum_{i=1}^m(\theta_i+\beta_i)} = C.$$

Now we consider the case $|x_B| > R$. By Lemma 5.2 we have that

$$\frac{|B|^{1+(1-\delta)/n}}{w^{-1}(B)} \lesssim R^{1-\delta}|x_B|^\alpha \lesssim R^{1-\delta+\alpha}, \tag{5.5}$$

because $\alpha < 0$. Since $|x_B| > R$, there exists a number $N \in \mathbb{N}$ such that $2^N R < |x_B| \leq 2^{N+1}R$. For $i \in \mathcal{I}_1$ we write

$$\begin{aligned}\left\| \frac{v_i^{-1}\mathcal{X}_{\mathbb{R}^n \setminus B}}{|x_B - \cdot|^{n-\gamma/m+1/m}} \right\|_\infty &\lesssim \sum_{k=0}^{N} \left\| \frac{v_i^{-1}\mathcal{X}_{B_{k+1}\setminus B_k}}{|x_B - \cdot|^{n-\gamma/m+1/m}} \right\|_\infty \\ &\quad + \sum_{k=N+1}^{\infty} \left\| \frac{v_i^{-1}\mathcal{X}_{B_{k+1}\setminus B_k}}{|x_B - \cdot|^{n-\gamma/m+1/m}} \right\|_\infty \\ &= S_1^i + S_2^i.\end{aligned}$$

By standard estimation we have that

$$S_1^i \lesssim |x_B|^{-\beta_i} \sum_{k=0}^{N} \left(2^k R\right)^{-n+\gamma/m-1/m} \lesssim |x_B|^{-\beta_i} R^{-n+\gamma/m-1/m} = |x_B|^{-\beta_i} R^{-\theta_i}$$

and

$$\begin{aligned} S_2^i &\lesssim \sum_{k=N+1}^{\infty} \left(2^k R\right)^{-\beta_i-n+\gamma/m-1/m} \\ &\lesssim \left(2^N R\right)^{-\beta_i-n+\gamma/m-1/m} \sum_{k=0}^{\infty} 2^{k(-\beta_i-n+\gamma/m-1/m)} \\ &\lesssim |x_B|^{-\beta_i} R^{-n+\gamma/m-1/m} = |x_B|^{-\beta_i} R^{-\theta_i}.\end{aligned}$$

These inequalities imply that

$$\prod_{i\in\mathcal{I}_1} \left\| \frac{v_i^{-1}\mathcal{X}_{\mathbb{R}^n \setminus B}}{|x_B - \cdot|^{n-\gamma/m+1/m}} \right\|_\infty \lesssim |x_B|^{-\sum_{i\in\mathcal{I}_1}\beta_i} R^{-\sum_{i\in\mathcal{I}_1}\theta_i}. \tag{5.6}$$

If $i \in \mathcal{I}_2$, we split the integral in a similar way to get

$$\begin{aligned}\left(\int_{\mathbb{R}^n\setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n-\gamma/m+1/m)p_i'}}\,dy\right)^{1/p_i'} &\lesssim \sum_{k=0}^{\infty} (2^k R)^{-n+\gamma/m-1/m} \left(\int_{B_k} |y|^{-\beta_i p_i'}\,dy\right)^{1/p_i'} \\ &= \sum_{k=0}^{N} + \sum_{k=N+1}^{\infty} \\ &= S_1^i + S_2^i.\end{aligned}$$

We shall estimate the sum $S_1^i + S_2^i$ by distinguishing into the cases $\theta_i < 0$, $\theta_i = 0$ and $\theta_i > 0$. Let us first assume that $\theta_i < 0$. Then by Lemma 5.2 we obtain

$$\begin{aligned} S_1^i &\lesssim \sum_{k=0}^{N} (2^k R)^{-n+\gamma/m-1/m+n/p_i'} |x_B|^{-\beta_i} \\ &\lesssim |x_B|^{-\beta_i} R^{-\theta_i} \sum_{k=0}^{N} 2^{-k\theta_i} \\ &\lesssim |x_B|^{-\beta_i} (2^N R)^{-\theta_i} \\ &\lesssim |x_B|^{-\beta_i-\theta_i}, \end{aligned}$$

since $\theta_i < 0$. For S_2^i we apply again Lemma 5.2 in order to get

$$\begin{aligned} S_2^i &\lesssim \sum_{k=N+1}^{\infty} (2^k R)^{-n+\gamma/m-1/m+n/p_i'-\beta_i} \\ &\lesssim \sum_{k=N+1}^{\infty} \left(2^k R\right)^{-\beta_i-\theta_i} \\ &= \left(2^{N+1} R\right)^{-\beta_i-\theta_i} \sum_{k=0}^{\infty} 2^{-k(\beta_i+\theta_i)} \\ &\lesssim |x_B|^{-\beta_i-\theta_i}, \end{aligned}$$

since $\theta_i + \beta_i > 0$. This yields

$$S_1^i + S_2^i \lesssim |x_B|^{-\beta_i-\theta_i} \tag{5.7}$$

when $\theta_i < 0$.

Now assume that $\theta_i = 0$. By proceeding similarly as in the previous case, we have

$$S_1^i \lesssim |x_B|^{-\beta_i} N \lesssim |x_B|^{-\beta_i} \log_2\left(\frac{|x_B|}{R}\right),$$

and

$$S_2^i \lesssim |x_B|^{-\beta_i}$$

since $\beta_i > 0$ when $\theta_i = 0$. Consequently,

$$S_1^i + S_2^i \lesssim |x_B|^{-\beta_i}\left(1 + \log_2\left(\frac{|x_B|}{R}\right)\right) \lesssim |x_B|^{-\beta_i} \log_2\left(\frac{|x_B|}{R}\right). \tag{5.8}$$

We finally consider the case $\theta_i > 0$. For S_2^i we can proceed exactly as in the case $\theta_i < 0$ and get the same bound. On the other hand, for S_1^i we have that

$$\begin{aligned}
S_1^i &\lesssim \sum_{k=0}^{N} (2^k R)^{-n+\gamma/m-1/m+n/p_i'} |x_B|^{-\beta_i} \\
&\lesssim |x_B|^{-\beta_i} R^{-\theta_i} \sum_{k=0}^{N} 2^{-k\theta_i} \\
&\lesssim |x_B|^{-\beta_i} \left(2^N R\right)^{-\theta_i} 2^{N\theta_i} \\
&\lesssim |x_B|^{-\beta_i-\theta_i} 2^{N\theta_i}.
\end{aligned}$$

Therefore, if $i \in \mathcal{I}_2$ and $\theta_i > 0$, we get

$$S_1^i + S_2^i \lesssim |x_B|^{-\beta_i-\theta_i} \left(1 + 2^{N\theta_i}\right) \lesssim 2^{N\theta_i} |x_B|^{-\beta_i-\theta_i}. \tag{5.9}$$

By combining (5.7),(5.8), and (5.9) we obtain

$$\begin{aligned}
&\prod_{i\in\mathcal{I}_2} \left(\int_{\mathbb{R}^n\setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n+\frac{1-\gamma}{m})p_i'}}\, dy \right)^{1/p_i'} \\
&\quad\lesssim \prod_{i\in\mathcal{I}_2,\theta_i<0} |x_B|^{-\beta_i-\theta_i} \prod_{i\in\mathcal{I}_2,\theta_i=0} |x_B|^{-\beta_i} \log_2\left(\frac{|x_B|}{R}\right) \prod_{i\in\mathcal{I}_2,\theta_i>0} |x_B|^{-\beta_i-\theta_i} 2^{N\theta_i} \\
&\quad\lesssim |x_B|^{-\sum_{i\in\mathcal{I}_2}(\beta_i+\theta_i)} 2^{N\sum_{i\in\mathcal{I}_2,\theta_i>0}\theta_i} \left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathcal{I}_2,\theta_i=0\}}.
\end{aligned}$$

The estimate above combined with (5.5) and (5.6) allows us to bound the left-hand side of (5.1) by

$$\begin{aligned}
&CR^{1-\delta+\alpha} |x_B|^{-\sum_{i\in\mathcal{I}_1}\beta_i} R^{-\sum_{i\in\mathcal{I}_1}\theta_i} |x_B|^{-\sum_{i\in\mathcal{I}_2}(\beta_i+\theta_i)} 2^{N\sum_{i\in\mathcal{I}_2,\theta_i>0}\theta_i} \\
&\qquad\times \left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathcal{I}_2,\theta_i=0\}}
\end{aligned}$$

or equivalently by

$$\left(\frac{R}{|x_B|}\right)^{1-\delta+\alpha-\sum_{i\in\mathcal{I}_1}\theta_i-\sum_{i\in\mathcal{I}_2,\theta_i>0}\theta_i} \left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathcal{I}_2,\theta_i=0\}}. \tag{5.10}$$

Since $\sum_{i=1}^m \theta_i = n/p + 1 - \gamma$ and $\alpha = \delta + \sum_{i=1}^m \beta_i + n/p - \gamma$, then the exponent of $R/|x_B|$ is equal to

$$\begin{aligned}
1-\delta+\alpha-\sum_{i\in\mathcal{I}_1}\theta_i - \sum_{i\in\mathcal{I}_2,\theta_i>0}\theta_i &= 1 + \sum_{i=1}^m \beta_i + n/p - \gamma - \sum_{i\in\mathcal{I}_1}\theta_i - \sum_{i\in\mathcal{I}_2,\theta_i>0}\theta_i \\
&= \sum_{i\in\mathcal{I}_2,\theta_i<0}(\beta_i+\theta_i) + \sum_{i\in\mathcal{I}_1}\beta_i + \sum_{i\in\mathcal{I}_2,\theta_i\geq 0}\beta_i \\
&= \nu - m_1\beta,
\end{aligned}$$

which is positive from our election of β. Since $\log t \lesssim \varepsilon^{-1}t^{\varepsilon}$ for every $t \geq 1$ and every $\varepsilon > 0$, we can bound (5.10) by

$$C\left(\frac{R}{|x_B|}\right)^{\nu-m_1\beta-\varepsilon\#\{i\in\mathcal{I}_2,\theta_i=0\}},$$

and this exponent is positive provided we choose $\varepsilon > 0$ sufficiently small. The proof is complete when $\mathcal{I}_1 \neq \emptyset$. Otherwise, we can follow the same steps and define the same parameters, omitting the factor corresponding to $\mathcal{I}_1$. This concludes the proof. □

We finish with the proof of the theorem dealing with the case $w = \prod_{i=1}^m v_i$.

Proof of Theorem 1.3. Let $\alpha = p/(mp-1)$ and assume that $\alpha > 1$. If $\vec{v} \in \mathcal{H}_m(\vec{p},\gamma,\delta)$, then by condition (2.3) we get

$$|B|^{-\delta/n+\gamma/n-1/p}\prod_{i\in\mathcal{I}_1}\|v_i^{-1}\mathcal{X}_B\|_\infty\prod_{i\in\mathcal{I}_2}\left(\frac{1}{|B|}\int_B v_i^{-p_i'}\right)^{1/p_i'} \leq \frac{C}{|B|}\int_B\prod_{i=1}^m v_i^{-1}. \tag{5.11}$$

Notice that $\sum_{i=1}^m \alpha/p_i' = 1$. Therefore we apply Hölder's inequality with p_i'/α in order to obtain

$$\left(\frac{1}{|B|}\int_B\left(\prod_{i=1}^m v_i^{-1}\right)^{\alpha}\right)^{1/\alpha} \leq \prod_{i\in\mathcal{I}_1}\|v_i^{-1}\mathcal{X}_B\|_\infty\prod_{i\in\mathcal{I}_2}\left(\frac{1}{|B|}\int_B v_i^{-p_i'}\right)^{1/p_i'}.$$

By multiplying each side of the inequality above by $|B|^{-\delta/n+\gamma/n-1/p}$ and using (5.11) we get

$$|B|^{-\delta/n+\gamma/n-1/p}\left(\frac{1}{|B|}\int_B\left(\prod_{i=1}^m v_i^{-1}\right)^{\alpha}\right)^{1/\alpha} \leq \frac{C}{|B|}\int_B\prod_{i=1}^m v_i^{-1}.$$

From this estimate we can conclude that

$$|B|^{-\delta/n+\gamma/n-1/p} \leq C$$

for every ball B, since $\alpha > 1$. Then we must have that $\delta/n = \gamma/n - 1/p$. □

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