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MIXED WEAK TYPE INEQUALITIES FOR THE FRACTIONAL MAXIMAL OPERATOR

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This paper is dedicated to the memory of Eleonor Harboure (Pola)

Abstract. Let $0 \le \alpha < n$, $q = \frac{n}{n-\alpha}$. We establish the mixed weak type inequality

$$\sup_{\lambda>0} \lambda^q \int_{\{x\in\mathbb{R}^n : M_\alpha f(x) > \lambda v(x)\}} u^q v^q \le C \left(\int_{\mathbb{R}^n} |f| u \right)^q$$

for the fractional maximal operator

$$M_\alpha f(x) = \sup_{h>0} |B(x,h)|^{\alpha/n-1} \int_{B(x,h)} |f|$$

under the following assumptions: (a) u^q belongs to the Muckenhoupt A_1 class, (b) v is essentially constant over dyadic annuli, and (c) $(\lambda v)^q$ satisfies a certain condition $C_p(\gamma)$ for all $\lambda > 0$. The last condition is fulfilled by any Muckenhoupt weight but it is also satisfied by some non Muckenhoupt weights. Our approach is based on the study of the same kind of inequalities for the local fractional maximal operator, a Hardy type operator and its adjoint.

1. Introduction

In this paper we study mixed weak type inequalities for the fractional maximal operator defined, for $0 \le \alpha < n$ and locally integrable functions $f : \mathbb{R}^n \to \mathbb{R}$, by

$$M_\alpha f(x) = \sup_{h>0} \frac{1}{|B(x,h)|^{1-\alpha/n}} \int_{B(x,h)} |f|,$$

where $B(x,h)$ is the euclidean ball centered at $x \in \mathbb{R}^n$ with radius h and $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. For $\alpha = 0$, M_α is the Hardy–Littlewood maximal operator M.

Mixed weak type inequalities for a sublinear operator T arise when studying weak type inequalities for the modified operator $Sf = \frac{T(fv)}{v}$, where v is a weight. By a weight we mean any positive not necessarily locally integrable function on $\mathbb{R}^n$.

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The expression mixed weak type inequality appeared in [1], although some years earlier estimates of this type had already been established in [18]. Later on, Sawyer studied this type of problems (see [24]). Since then, mixed weak type inequalities gained great interest, and they are also known as mixed weak estimates of Sawyer type. Among the papers about mixed weak type inequalities we highlight [14], [8], [6], [10], [22], [2], [3], [5], [11], [19], [20], [21] and [4].

The good weights for M_α were characterized in [17],[27] and [26]. In particular, if $u^q \in A_1$, for $q = \frac{n}{n-\alpha}$, that is $Mu^q \leq Cu^q$ a.e., then

$$\int_{\{x\in\mathbb{R}^n : M_\alpha f(x)>\lambda\}} u^q \leq \frac{C}{\lambda^q}\left(\int_{\mathbb{R}^n} |f|u\right)^q, \tag{1.1}$$

where C is independent of f and $\lambda > 0$ (the classes of A_p weights that appear in this theorem and in the following are the classical Muckenhoupt weights (see [16] and [9])). Recently, in [4], the authors explain the interest of studying the modified operator $Sf = \frac{M_\alpha(fv)}{v}$, i.e., the interest of studying mixed weak type inequalities for M_α. The following result can be found in [4] and [23].

Theorem 1.1 ([4], [23])**.** *Let $0 < \alpha < n$ and $q = \frac{n}{n-\alpha}$. If u, v are weights such that $u^q \in A_1$ and $v^q \in A_1$ or, more generally, $u^q \in A_1$ and $v^q \in A_\infty$, then the following estimate holds*

$$\int_{\{x\in\mathbb{R}^n : M_\alpha f(x)>\lambda v(x)\}} u^q v^q \leq \frac{C}{\lambda^q}\left(\int_{\mathbb{R}^n} |f|u\right)^q, \tag{1.2}$$

where C is independent of f and $\lambda > 0$.

Remark 1.2. *The case $u^q \in A_1$ and $v^q \in A_1$ was established in* [4] *while the general case $u^q \in A_1$ and $v^q \in A_\infty$ was stated in* [23]*. The theorem for the Hardy–Littlewood maximal operator was proved in* [10] *in the general case and in* [24] *when both weights are in A_1.*

The goal of this paper is to show that (1.2) holds under other conditions on v. These conditions provide pairs of weights for which (1.2) holds but $v^q \notin A_\infty$. In order to state our result, we introduce some definitions.

Definition 1.3. *We will say that a weight v is essentially constant over dyadic annuli if there exists a constant $C_v > 0$ such that*

$$\sup_{x\in C_k^{k+1}} v(x) \leq C_v \inf_{x\in C_k^{k+1}} v(x) \quad \textit{for all } k \in \mathbb{Z},$$

where $C_j^k = \{x \in \mathbb{R}^n : 2^j \leq |x| < 2^k\} = B(0,2^k) \setminus B(0,2^j)$ and sup *and* inf *stand for the essential supremum and essential infimum, respectively.*

Definition 1.4. *Let $1 < p < \infty$ and $\gamma \in (0,1)$. We say that a weight w satisfies condition $C_p(\gamma)$ if*

$$\|w\|_{C_p(\gamma)} = \sup_{0<b} b^{n\gamma}\left(\int_{B(0,b)\cap E} w^{1+\gamma}\right)^{1/p}\left(\int_{(\mathbb{R}^n\setminus B(0,b))\cap E} w^{1+\gamma}\right)^{1/p'} < \infty,$$

where $E = \{x \in \mathbb{R}^n : 0 < |x|^n < \frac{1}{w(x)}\}$.

Now we can state our main result.

Theorem 1.5. *Let $0 \le \alpha < n$, $q = \frac{n}{n-\alpha}$, and let u and v be two weights such that*

(a) $u^q \in A_1$,
(b) *v is essentially constant over dyadic annuli and*
(c) *$(\lambda v)^q$ satisfies $C_p(\gamma)$ for some $p > 1$ an some $\gamma \in (0,1)$ with $\|(\lambda v)^q\|_{C_p(\gamma)}$ independent of $\lambda > 0$.*

Then, there exists a constant $C > 0$ such that for all locally integrable functions f

$$\sup_{\lambda>0} \lambda^q \int_{\{x\in\mathbb{R}^n : M_\alpha f(x) > \lambda v(x)\}} u^q v^q \le C \left(\int_{\mathbb{R}^n} |f| u \right)^q . \tag{1.3}$$

Weights u satisfying assumption (a) are well known. However, it is natural to ask about examples of weights v which satisfy (b) and (c). In Section 3 of this paper we show that if v^q is a Muckenhoupt weight, that is, if v^q is an A_∞ weight then v^q satisfies (c). Therefore, the following theorem holds.

Theorem 1.6. *Let $0 \le \alpha < n$, $q = \frac{n}{n-\alpha}$, and let u and v be two weights such that $u^q \in A_1$, $v^q \in A_\infty$ and v is essentially constant over dyadic annuli. Then* (1.3) *holds.*

This result is weaker than Theorem 1.1 since v is additionally required to be essentially constant over dyadic annuli. However, we believe that, in some sense, the proof is simpler. Perhaps, the main strength of Theorem 1.5 is that assumption (c) is satisfied by weights which are not in A_∞. For instance, in Section 3 we prove that if $v(x)^q = \frac{h(|x|)}{|x|^{n\delta}}$, where $h : (0,\infty) \to [0,\infty)$ is a nonincreasing function and $\delta > 1$, then v satisfies (c). Therefore, we have the following theorem.

Theorem 1.7. *Let $0 \le \alpha < n$, $q = \frac{n}{n-\alpha}$ and $\delta \ne 1$. Let u, v be two weights such that $u^q \in A_1$ and $v(x)^q = \frac{1}{|x|^{n\delta}}$. Then* (1.3) *holds.*

Notice that the case $\delta < 1$ is contained in Theorem 1.6. We remark that the weights $v(x)^q = \frac{1}{|x|^{n\delta}}$, $\delta > 1$, are not A_∞ weights. Finally, we point out that the theorem does not hold if $\delta = 1$ (see [18]).

In order to prove Theorem 1.5 we split the fractional maximal operator into its local part and its global part, that is, we consider the following operators.

(1) The local operator

$$M_\alpha^{loc} f(x) = \sup_{0<h\le\frac{|x|}{2}} \frac{1}{|B(x,h)|^{1-\alpha/n}} \int_{B(x,h)} |f|,$$

and its complement

$$M_\alpha^{noloc} f(x) = \sup_{h>\frac{|x|}{2}} \frac{1}{|B(x,h)|^{1-\alpha/n}} \int_{B(x,h)} |f|.$$

It is clear that $M_\alpha f(x) = \max\{M_\alpha^{loc} f(x), M_\alpha^{noloc} f(x)\}$. Therefore, we reduce the study of M_α to the study of M_α^{loc} and M_α^{noloc}. We can estimate the second operator by using Hardy operators.

(2) The Hardy type operator and its adjoint are defined, respectively, by

$$T_\alpha f(x) = \frac{1}{|x|^{n-\alpha}} \int_{B(0,|x|)} f(y)\,dy, \quad \text{and} \quad T_\alpha^* f(x) = \int_{\mathbb{R}^n \setminus B(0,|x|)} \frac{f(y)}{|y|^{n-\alpha}}\,dy.$$

For $\alpha = 0$ we just write T or T^*.

Let us see that M_α^{noloc} is controlled by T_α and T_α^*. In order to see this, let $\beta := \frac{1}{q} = 1 - \alpha/n$. Then, if we fix $x \in \mathbb{R}^n$ and $h > \frac{|x|}{2}$, we have that

$$\begin{aligned}\frac{1}{|B(x,h)|^\beta} \int_{B(x,h)} |f| &\le \frac{1}{|B(x,h)|^\beta} \int_{B(0,|x|+h)} |f| \\ &= \frac{1}{|B(x,h)|^\beta} \int_{B(0,|x|)} |f| + \frac{1}{|B(x,h)|^\beta} \int_{B(0,|x|+h)\setminus \overline{B}(0,|x|)} |f| \\ &= I + II.\end{aligned}$$

Observe that $h > \frac{|x|}{2}$ implies that $|B(x,h)|^\beta = |B_1|^\beta h^{n-\alpha} > |B_1|^\beta \left(\frac{|x|}{2}\right)^{n-\alpha}$, where $B_1 = B(0,1)$ denotes the unit ball in $\mathbb{R}^n$. Then

$$I \le \frac{2^{n-\alpha}}{|B_1|^\beta} T_\alpha |f|(x).$$

In order to estimate II, let $y \in B(0,|x|+h) \setminus B(0,|x|)$ with $h > \frac{|x|}{2}$. Then, $|x| \le |y| < |x| + h < 3h$ implies $h > \frac{|y|}{3}$, which gives $|B(x,h)| = |B_1| h^n > |B_1| \left(\frac{|y|}{3}\right)^n$. Therefore

$$II \le \frac{3^{n-\alpha}}{|B_1|^\beta} \int_{B(0,|x|+h)\setminus B(0,|x|)} \frac{|f(y)|}{|y|^{n-\alpha}} dy \le \frac{3^{n-\alpha}}{|B_1|^\beta} T_\alpha^* |f|(x).$$

Taking into account these estimates, we obtain

$$M_\alpha f(x) \le \max\left\{ M_\alpha^{loc} f(x), \frac{2^{n-\alpha}}{|B_1|^\beta} T_\alpha |f|(x) + \frac{3^{n-\alpha}}{|B_1|^\beta} T_\alpha^* |f|(x) \right\}. \tag{1.4}$$

It follows from this inequality that we can obtain weighted mixed weak type inequalities for M_α from the corresponding ones for the operators M_α^{loc}, T_α and T_α^*. In particular, we shall prove Theorem 1.5 in this way.

The remainder of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.5 (we establish the estimates for M_α^{loc}, T_α^* and T_α and finally we gather them and prove Theorem 1.5), and Section 3 contains the results providing examples of weights which satisfy the assumptions of Theorem 1.5.

2. Proof of Theorem 1.5

The mixed weak type inequalities for M_α^{loc} are very easy to handle under the assumption that v is essentially constant over dyadic annuli. For T_α^* it is even simpler since we do not need to assume anything on v and, furthermore, the strong type inequality holds. In fact, in both cases, the inequalities follow from the corresponding estimate for $v = 1$. We write the proofs here for the sake of completeness.

2.1. **Weighted mixed weak type inequalities for M_α^{loc}.** Let us recall that a weight w belongs to the A_1 class of Muckenhoupt if there exists a constant C such that

$$Mw(x) \leq C_w w(x) \quad \text{for almost every } x.$$

The smallest possible C in the above inequality is denoted by $[w]_{A_1}$. Here and in what follows, C_w denotes a constant depending only on the weight and on the dimension but independent of other parameters.

Theorem 2.1. *Let $0 \leq \alpha < n$ and $q = \frac{n}{n-\alpha}$. Let u, v be two weights such that $u^q \in A_1$ and v is essentially constant over dyadic annuli. Then there exists a constant $C > 0$ such that for all $f \in L^1_{loc}(\mathbb{R}^n)$*

$$\sup_{\lambda>0} \lambda^q \int_{\{x\in\mathbb{R}^n : M_\alpha^{loc} f(x) > \lambda v(x)\}} u^q v^q \leq C \left(\int_{\mathbb{R}^n} |f| u \right)^q .$$

Proof. It is enough to prove the inequality for $\lambda = 1$. Let $k \in \mathbb{Z}$,

$$\alpha_k = \sup\{v(x) : x \in C_k^{k+1}\} \quad \text{and} \quad \beta_k = \inf\{v(x) : x \in C_k^{k+1}\}.$$

Then, for all $x \in C_k^{k+1}$, we have that

$$M_\alpha^{loc} f(x) = M_\alpha^{loc} \left(f\chi_{C_{k-1}^{k+2}} \right)(x) \leq M_\alpha \left(f\chi_{C_{k-1}^{k+2}} \right)(x).$$

Therefore, using that v is essentially constant over dyadic annuli and (1.1) (since $u^q \in A_1$), we obtain

$$\begin{aligned} &\int_{\{x\in\mathbb{R}^n : M_\alpha^{loc} f(x) > v(x)\}} u^q v^q \leq \sum_{k\in\mathbb{Z}} \int_{\{x\in C_k^{k+1} : M_\alpha^{loc} f(x) > v(x)\}} u^q v^q \\ &\leq \sum_{k\in\mathbb{Z}} \alpha_k^q \int_{\{x\in C_k^{k+1} : M_\alpha^{loc} f(x) > v(x)\}} u^q \leq \sum_{k\in\mathbb{Z}} \alpha_k^q \int_{\{x\in C_k^{k+1} : M_\alpha \left(f\chi_{C_{k-1}^{k+2}} \right)(x) > \beta_k\}} u^q \\ &\leq C_u \sum_{k\in\mathbb{Z}} \left(\frac{\alpha_k}{\beta_k} \right)^q \left(\int_{C_{k-1}^{k+2}} |f| u \right)^q \leq C_{u,v} \left(\int_{\mathbb{R}^n} |f| u \right)^q . \end{aligned}$$

□

2.2. **Weighted mixed weak type inequalities for T_α^*.** In order to state the mixed inequality for T_α^* we introduce some definitions.

Definition 2.2. *Let $1 \le p < \infty$. We say that a weight ω belongs to the class $A_p(T^*)$ if*

$$\|w\|_{A_p(T^*)} = \sup_{b>0} \left(\int_{B(0,b)} w \right)^{1/p} \left(\int_{\mathbb{R}^n \setminus B(0,b)} \frac{w(x)^{1-p'}}{|x|^{np'}} \, dx \right)^{1/p'} < \infty.$$

When $p = 1$, the second bracket has the usual meaning, i.e., $\sup_{|x|>b} |x|^{-n} w^{-1}(x)$. We notice that ω belongs to the class $A_1(T^*)$ if and only if there exists a constant $C_\omega > 0$ such that, for a.e. $y \in \mathbb{R}^n$,

$$\frac{1}{|y|^n} \int_{B(0,|y|)} \omega \le C_\omega \omega(y).$$

It is known that $w \in A_p(T^*)$ if and only if T^* is bounded in $L^p(w)$ (see [7]). It also holds that $A_p \subset A_p(T^*)$. It is obvious for $p = 1$. To prove the inclusion when $p > 1$, let $b > 0$ and consider $f = w\chi_{B(0,b)}$. Then $Mf(x) \ge \frac{C}{|x|^n} \int_{B(0,b)} w$ for all $x \in \mathbb{R}^n \setminus B(0,b)$. Now we use that $w \in A_p \Leftrightarrow w^{1-p'} \in A_{p'}$ and then M is bounded in $L^{p'}(w^{1-p'})$ to obtain that

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(0,b)} \frac{w(x)^{1-p'}}{|x|^{np'}} \, dx &\le C \int_{\mathbb{R}^n} (Mf(x))^{p'} w(x)^{1-p'} \, dx \left(\int_{B(0,b)} w \right)^{-p'} \\ &\le C \left(\int_{B(0,b)} w \right)^{1-p'}. \end{aligned}$$

Theorem 2.3. *Let $0 \le \alpha < n$ and $q = \frac{n}{n-\alpha}$. Let u be a weight such that $u^q \in A_1(T^*)$. Then, there exists $C > 0$ such that for every weight v, every $\lambda > 0$ and all locally integrable functions f,*

$$\int_{\{x \in \mathbb{R}^n : |T^*_\alpha f(x)| > \lambda v(x)\}} u^q v^q \le \frac{C}{\lambda^q} \left(\int_{\mathbb{R}^n} |f| u \right)^q.$$

Furthermore,

$$\int_{\mathbb{R}^n} |T^*_\alpha f|^q u^q \le C \left(\int_{\mathbb{R}^n} |f| u \right)^q.$$

Proof. The strong inequality follows by the Minkowski inequality, the fact that $q(n-\alpha) = n$ and $u^q \in A_1(T^*)$:

$$\begin{aligned} \int_{\mathbb{R}^n} |T^*_\alpha f|^q u^q &\le \left(\int_{\mathbb{R}^n} |f(y)| \left(\frac{1}{|y|^n} \int_{B(0,|y|)} u^q(x) dx \right)^{1/q} dy \right)^q \\ &\le C_u \left(\int_{\mathbb{R}^n} |f(y)| u(y) dy \right)^q. \end{aligned}$$

□

2.3. **Weighted mixed weak type inequalities for T_α.** In the next result we characterize the mixed weak type inequality for the operator T_α. Our argument is strongly based on the paper [15].

Proposition 2.4. *Let $0 \le \alpha < n$ and $\beta = \frac{1}{q} = \frac{n-\alpha}{n}$. The following assertions are equivalent:*

(1) *There exists a constant $C > 0$ such that, for all $f \in L^1_{loc}(\mathbb{R}^n)$,*

$$\sup_{\lambda>0} \lambda^q \int_{\{x\in\mathbb{R}^n : |T_\alpha f(x)|>\lambda v(x)\}} u^q v^q \le C \left(\int_{\mathbb{R}^n} |f| u \right)^q.$$

(2) *There exists a constant $\widetilde{C} > 0$ such that, for all real numbers $a > 0$,*

$$\sup_{\lambda>0} \lambda^q \int_{\left\{ |x|>a : \lambda v(x) < \frac{1}{|x|^{n\beta}} \right\}} u^q v^q \le \widetilde{C} \operatorname*{ess\,inf}_{x\in B(0,a)} u(x)^q. \tag{2.5}$$

Proof. Suppose that (1) holds. Let $a > 0$ and let $E \subset B(0,a)$ be a Lebesgue measurable set such that $|E| > 0$. Let us consider $f = \frac{1}{|E|}\chi_E$. Then, for all x with $|x| > a$,

$$T_\alpha f(x) = \frac{1}{|x|^{n\beta}} \int_{B(0,|x|)} f(y)dy = \frac{1}{|x|^{n\beta}}.$$

Therefore, using (1), for all $\lambda > 0$, we obtain

$$\begin{aligned} \int_{\left\{ |x|>a : \lambda v(x) < \frac{1}{|x|^{n\beta}} \right\}} u^q v^q &\le \int_{\{|x|>a : |T_\alpha f(x)|>\lambda v(x)\}} u^q v^q \\ &\le \frac{C}{\lambda^q} \left(\int_{\mathbb{R}^n} |f| u \right)^q = \frac{C}{\lambda^q} \left(\frac{1}{|E|} \int_E u \right)^q. \end{aligned}$$

Since the above inequality holds for any measurable set $E \subset B(0,a)$ with $|E| > 0$, using the Lebesgue differentiation theorem we obtain (2).

In order to prove that (2) implies (1) we can assume that $f \ge 0$, $f \in L^1(\mathbb{R}^n)$ and $\int_{B(0,a)} f > 0$, for all $a > 0$.

Let $\{x_k\}_{k=0}^\infty$ be a sequence of numbers defined recursively as follows: $x_0 = \infty$, and if x_k has been chosen, we take x_{k+1} such that

$$\int_{\{x :\, |x| < x_{k+1}\}} f = \int_{\{x :\, x_{k+1} \le |x| < x_k\}} f.$$

Then, given x with $x_{k+1} \le |x| < x_k$,

$$\begin{aligned} T_\alpha f(x) = \frac{1}{|x|^{n\beta}} \int_{B(0,|x|)} f(y)dy &\le \frac{1}{|x|^{n\beta}} \int_{B(0,x_k)} f(y)dy \\ = \frac{4}{|x|^{n\beta}} \int_{\{y :\, x_{k+2} \le |y| < x_{k+1}\}} f. \end{aligned}$$

Therefore, the set $\{x_{k+1} \le |x| < x_k : T_\alpha f(x) > v(x)\}$ is a subset of

$$\left\{ x_{k+1} \le |x| < x_k : \frac{1}{|x|^{n\beta}} > \frac{v(x)}{4 \int_{\{y :\, x_{k+2} \le |y| < x_{k+1}\}} f} \right\}.$$

For each $k \in \mathbb{N}$, let

$$\beta_k = \operatorname{ess\,inf}\{u(y)^q : y \in B(0, x_{k+1})\} \text{ and } \lambda_k = \left(4 \int_{\{y:\, x_{k+2} \le |y| < x_{k+1}\}} f\right)^{-1}.$$

Then, using (2), we have

$$\beta_k > \frac{\lambda_k^q}{\widetilde{C}} \int_{\left\{|x| > x_{k+1} : \lambda_k v(x) < \frac{1}{|x|^{n\beta}}\right\}} u^q v^q.$$

Therefore

$$\begin{aligned}
\int_{\{x: T_\alpha f(x) > v(x)\}} u^q v^q &\le \sum_{k=0}^{\infty} \int_{\{x_{k+1} \le |x| < x_k : T_\alpha f(x) > v(x)\}} u^q v^q \\
&\le \sum_{k=0}^{\infty} \int_{\left\{x_{k+1} \le |x| < x_k : \lambda_k v(x) < \frac{1}{|x|^{n\beta}}\right\}} u^q v^q \\
&\le 4^q \widetilde{C} \sum_{k=0}^{\infty} \left(\int_{\{x_{k+2} \le |x| < x_{k+1}\}} f \beta_k^{1/q} \right)^q \\
&\le 4^q \widetilde{C} \sum_{k=0}^{\infty} \left(\int_{\{x_{k+2} \le |x| < x_{k+1}\}} f u \right)^q \le C \left(\int_{\mathbb{R}^n} f u \right)^q. \qquad \square
\end{aligned}$$

Observe that in the particular case of $v = 1$ we obtain a characterization of the good weights for the weak type $(1, q)$ inequality of T_α.

Corollary 2.5. *Let $0 \le \alpha < n$ and $q = \frac{n}{n-\alpha}$. The following assertions are equivalent:*

(1) *There exists a constant $C > 0$ such that, for all $f \in L^1_{loc}(\mathbb{R}^n)$,*

$$\sup_{\lambda > 0} \lambda^q \int_{\{x \in \mathbb{R}^n : |T_\alpha f(x)| > \lambda\}} u^q \le C \left(\int_{\mathbb{R}^n} |f| u \right)^q. \tag{2.6}$$

(2) *There exists a constant $\widetilde{C} > 0$ such that, for all real numbers $a > 0$,*

$$\sup_{b > a} \frac{1}{b^n} \int_{\{a < |y| < b\}} u^q \le \widetilde{C} \operatorname*{ess\,inf}_{x \in B(0,a)} u(x)^q. \tag{2.7}$$

If a weight $\omega = u^q$ satisfies condition (2.7), we say that $w \in A_1(T)$. From the definitions it follows immediately that $A_1 \subset A_1(T)$.

By using this characterization we obtain sufficient conditions on u and v so that (2.6) holds.

Theorem 2.6. *Let $0 \leq \alpha < n$, $q = \frac{n}{n-\alpha}$, and let u and v be two weights such that, $u^{q(1+\varepsilon)} \in A_1(T)$ and, for all $\lambda > 0$, $(\lambda v)^q$ satisfies $C_p(\gamma)$ for some $0 < \gamma < \varepsilon < 1 < p$, with constant independent on λ. Then, there exists a constant $C > 0$ such that, for all $f \in L^1_{loc}(\mathbb{R}^n)$,*

$$\sup_{\lambda>0} \lambda^q \int_{\{x\in\mathbb{R}^n:|T_\alpha f(x)|>\lambda v(x)\}} u^q v^q \leq C \left(\int_{\mathbb{R}^n} |f|u \right)^q .$$

Proof. By Proposition 2.4, we only have to prove condition (2.5). We can assume that $\lambda = 1$ since changing v by λv we obtain the result (here we are using that $(\lambda v)^q$ satisfies $C_p(\gamma)$ for some $0 < \gamma < \varepsilon < 1 < p$, with constant independent on λ).

Fix $a > 0$, consider the set $E_a = \{x : |x| > a \text{ and } v(x) < \frac{1}{|x|^{n\beta}}\}$, where, as before, $\beta = 1/q$. Also, we can assume that $|E_a| > 0$. Denote $z = \operatorname{ess\,sup}\{|x| : x \in E_a\}$. Then,

$$\int_{\left\{a<|x|<z : v(x)<\frac{1}{|x|^{n\beta}}\right\}} v^{q(1+\gamma)} \leq \int_{\mathbb{R}^n\setminus B(0,a)} \frac{1}{|x|^{n(1+\gamma)}}\, dx < \infty.$$

Therefore, we can define a sequence, $\{z_n\}_{n=0}^{\infty}$, as follows: $z_0 = a$ and, if z_k has been chosen, we take z_{k+1} such that

$$\int_{\{z_{k+1}<|x|<z\}\cap E_a} v^{q(1+\gamma)} = \int_{\{z_k<|x|\leq z_{k+1}\}\cap E_a} v^{q(1+\gamma)}.$$

This implies that

$$\int_{\{a<|x|\leq z_1\}\cap E_a} v^{q(1+\gamma)} = 2^k \int_{\{z_{k+1}<|x|<z\}\cap E_a} v^{q(1+\gamma)}$$

and $\lim_{n\to\infty} z_n = z$. As a consequence of the $C_p(\gamma)$ condition,

$$z_{k+1}^{n\gamma} \left(\int_{\{a<|x|\leq z_1\}\cap E_a} v^{q(1+\gamma)} \right)^{1/p} \left(\int_{\{z_{k+1}<|x|<z\}\cap E_a} v^{q(1+\gamma)} \right)^{1/p'} \leq C.$$

Then,

$$2^{k/p} z_{k+1}^{n\gamma} \int_{\{z_{k+1}<|x|<z\}\cap E_a} v^{q(1+\gamma)} \leq C$$

and, by the definition of z_k,

$$\int_{\{z_k<|x|\leq z_{k+1}\}\cap E_a} v^{q(1+\gamma)} \leq \frac{C}{2^{k/p} z_{k+1}^{n\gamma}}. \tag{2.8}$$

Notice that,

$$\int_{E_a} u^q v^q = \int_{\{a<|x|\leq z\}\cap E_a} u^q v^q = \sum_{k=0}^{\infty} \int_{\{z_k<|x|\leq z_{k+1}\}\cap E_a} u^q v^q. \tag{2.9}$$

Now, let us estimate each term of this sum. Observe that since $1 - \gamma\varepsilon > 0$, for all $x \in E_a$,

$$v(x)^{q\left(\frac{1-\gamma\varepsilon}{1+\varepsilon}\right)} < \frac{1}{|x|^{n\left(\frac{1-\gamma\varepsilon}{1+\varepsilon}\right)}}.$$

Using this, (2.8) and Hölder inequality we obtain

$$\begin{aligned}\int_{\{z_k<|x|\le z_{k+1}\}\cap E_a} u^q v^q &\le \int_{\{z_k<|x|\le z_{k+1}\}\cap E_a} u(x)^q v(x)^{q\left(\frac{(1+\gamma)\varepsilon}{1+\varepsilon}\right)} \frac{1}{|x|^{n\left(\frac{1-\gamma\varepsilon}{1+\varepsilon}\right)}}\,dx \\ &\le \left(\int_{\{z_k<|x|\le z_{k+1}\}\cap E_a} \frac{u(x)^{q(1+\varepsilon)}}{|x|^{n(1-\gamma\varepsilon)}}\,dx\right)^{\frac{1}{1+\varepsilon}} \left(\int_{\{z_k<|x|\le z_{k+1}\}\cap E_a} v(x)^{q(1+\gamma)}\,dx\right)^{\frac{\varepsilon}{1+\varepsilon}} \\ &\le C\left(\int_{\{a<|x|\le z_{k+1}\}} \frac{u(x)^{q(1+\varepsilon)}}{|x|^{n(1-\gamma\varepsilon)}}\,dx\right)^{\frac{1}{1+\varepsilon}} \left(\frac{1}{2^{k/p} z_{k+1}^{n\gamma}}\right)^{\frac{\varepsilon}{1+\varepsilon}}. \end{aligned} \tag{2.10}$$

Now we fix $\delta > 1$ and let $b \in B(0,a)$ be a point of the Lebesgue set for $u^{q(1+\varepsilon)}$ such that $u(b) \le \delta \operatorname{ess\,inf}_{x\in B(0,a)} u(x)$. Let us take r with $0 < r < a - |b|$. This implies that $B(b,r) \subset B(0,a)$ and therefore, taking $f = \chi_{B(b,r)}$ we have that, for all $x \in \mathbb{R}^n$, $|x| > a$,

$$Tf(x) = \frac{|B(b,r)|}{|x|^n}.$$

We can now use Kolmogorov's inequality, since $u^{q(1+\varepsilon)} \in A_1(T)$ and then T applies $L^1(u^{q(1+\varepsilon)})$ into weak-$L^1(u^{q(1+\varepsilon)})$. Then

$$\begin{aligned}\int_{\{a<|x|\le z_{k+1}\}} & \frac{u(x)^{q(1+\varepsilon)}}{|x|^{n(1-\gamma\varepsilon)}}\,dx \\ &= \frac{1}{|B(b,r)|^{1-\gamma\varepsilon}} \int_{\{a<|x|\le z_{k+1}\}} |Tf(x)|^{1-\gamma\varepsilon} u(x)^{q(1+\varepsilon)}\,dx \\ &\le \left(\int_{\{a<|x|\le z_{k+1}\}} u^{q(1+\varepsilon)}\right)^{\gamma\varepsilon} \left(\frac{1}{|B(b,r)|}\int_{B(b,r)} u^{q(1+\varepsilon)}\right)^{1-\gamma\varepsilon}. \end{aligned} \tag{2.11}$$

Using the definition of $u^{q(1+\varepsilon)} \in A_1(T)$, we obtain that the inequality above is bounded by

$$\le C z_{k+1}^{n\gamma\varepsilon} \left(\operatorname*{ess\,inf}_{x\in B(0,a)} u(x)\right)^{q(1+\varepsilon)\gamma\varepsilon} \left(\frac{1}{|B(b,r)|}\int_{B(b,r)} u^{q(1+\varepsilon)}\right)^{1-\gamma\varepsilon}. \tag{2.12}$$

Recall that we have to prove condition (2.5). Then, putting together inequalities (2.9), (2.10), (2.11) and (2.12), letting r tend to 0 and using the Lebesgue differentiation theorem and the condition on $b \in B(0,a)$, we obtain

$$\int_{E_a} u^q v^q \le C\delta^{q(1-\gamma\varepsilon)} \operatorname*{ess\,inf}_{x\in B(0,a)} u(x)^q \sum_{k=0}^{\infty} \left(\frac{1}{2^{\frac{\varepsilon}{p(1+\varepsilon)}}}\right)^k.$$

Finally, letting δ tend to 1, we obtain the desired inequality (2.5). □

Following the same steps as in Proposition 2.3 of [12], we settle the following result.

Proposition 2.7. *Let* $p > 1$ *and* $\gamma \in (0,1)$. *If* w *is a weight such that* $w^{1+\gamma} \in A_p(T^*)$ *then* $\lambda w \in C_p(\gamma)$ *for all* $\lambda > 0$ *and*

$$\|\lambda w\|_{C_p(\gamma)} \leq \|w^{1+\gamma}\|_{A_p(T^*)}.$$

This allows us to obtain the following corollary.

Corollary 2.8. *Let* $0 \leq \alpha < n$, $q = \frac{n}{n-\alpha}$, *and let* u *and* v *be two weights such that* $u^q \in A_1$ *and* $v^q \in A_\infty$. *Then, there exists a constant* $C > 0$ *such that, for all* $f \in L^1_{loc}(\mathbb{R}^n)$,

$$\sup_{\lambda>0} \lambda^q \int_{\{x\in\mathbb{R}^n : |T_\alpha f(x)| > \lambda v(x)\}} u^q v^q \leq C \left(\int_{\mathbb{R}^n} |f| u \right)^q.$$

Proof. Since $u^q \in A_1$ then there exists $\varepsilon > 0$ as small as we need such that $u^{q(1+\varepsilon)} \in A_1 \subset A_1(T)$. On the other hand, $v^q \in A_\infty$ implies that there exists $p > 1$ such that $v^q \in A_p$ and then there exists $\gamma \in (0, \varepsilon)$ satisfying that $v^{q(1+\gamma)} \in A_p \subset A_p(T^*)$. Therefore, the corollary follows from Proposition 2.7 and Theorem 2.6. □

2.4. **Proof of Theorem 1.5.** The proof of Theorem 1.5 follows immediately from inequality (1.4) and Theorems 2.1, 2.3 and 2.6.

3. Some examples of weights in $C_p(\gamma)$

In this section we give some examples of weights v such that $(\lambda v)^q \in C_p(\gamma)$ with constant independent of $\lambda > 0$, for $p > 1$ and some $\gamma \in (0,1)$.

Proposition 3.1. *Let* $h : (0,\infty) \to (0,\infty)$, $\delta > 0$ *and* $v(x)^q = \frac{h(|x|)}{|x|^{n\delta}}$.

(1) *If* h *is decreasing and* $\delta > 1$ *then, for all* $p > 1$, *there exists* $\gamma \in (0,1)$ *such that* $(\lambda v)^q \in C_p(\gamma)$ *with constant independent on* $\lambda > 0$.
(2) *If* h *is increasing and* $\delta < 1$ *then there exists* $\gamma \in (0,1)$ *such that* $(\lambda v)^q \in C_p(\gamma)$ *with constant independent on* $\lambda > 0$.

Proof. (1) We have to prove that there exists $\gamma \in (0,1)$ such that

$$\sup_{0<b} b^{n\gamma} \lambda^{q(1+\gamma)} \left(\int_{B(0,b)\cap E} \left(\frac{h(|x|)}{|x|^{n\delta}} \right)^{1+\gamma} dx \right)^{1/p} \left(\int_{(\mathbb{R}^n \setminus B(0,b))\cap E} \left(\frac{h(|x|)}{|x|^{n\delta}} \right)^{1+\gamma} dx \right)^{1/p'} < \infty, \tag{3.1}$$

where

$$E = \left\{ x \in \mathbb{R}^n : 0 < |x|^n < \frac{1}{(\lambda v(x))^q} \right\} = \left\{ x \in \mathbb{R}^n : 0 < |x|^n < \frac{|x|^{n\delta}}{\lambda^q h(|x|)} \right\}$$
$$= \left\{ x \in \mathbb{R}^n : 0 < |x| \text{ and } h(|x|) < \frac{|x|^{n(\delta-1)}}{\lambda^q} \right\}.$$

We can assume that $B(0,b) \cap E \neq \emptyset$. Since h is decreasing and $\lim_{|x|\to\infty} \frac{|x|^{n(\delta-1)}}{\lambda^q} = \infty$, there exists $\mu \in [0,b)$ such that $E = \{x \in \mathbb{R}^n : \mu \leq |x|\}$. If $\mu = 0$ then $h \equiv 0$

and there is nothing to prove. If $\mu > 0$ then, for all $x \in E$, $h(|x|) < \frac{\mu^{n(\delta-1)}}{\lambda^q}$. Therefore

$$\int_{B(0,b)\cap E} \left(\frac{h(|x|)}{|x|^{n\delta}}\right)^{1+\gamma} dx \leq \frac{\mu^{n(\delta-1)(1+\gamma)}}{\lambda^{q(1+\gamma)}} C \int_\mu^b r^{n-1} \frac{1}{r^{n\delta(1+\gamma)}}\, dr$$
$$\leq C \frac{\mu^{n(\delta-1)(1+\gamma)}}{\lambda^{q(1+\gamma)}} \frac{1}{n(\delta(1+\gamma)-1)} \frac{1}{\mu^{n(\delta(1+\gamma)-1)}}.$$

On the other hand,

$$\int_{(\mathbb{R}^n\setminus B(0,b))\cap E} \left(\frac{h(|x|)}{|x|^{n\delta}}\right)^{1+\gamma} dx \leq C \frac{\mu^{n(\delta-1)(1+\gamma)}}{\lambda^{q(1+\gamma)}} \int_b^\infty r^{n-1} \frac{1}{r^{n\delta(1+\gamma)}}\, dr$$
$$= C \frac{\mu^{n(\delta-1)(1+\gamma)}}{\lambda^{q(1+\gamma)}} \frac{1}{n(\delta(1+\gamma)-1)} \frac{1}{b^{n(\delta(1+\gamma)-1)}}.$$

Then, by inequalities 3 and 3,

$$b^{n\gamma}\lambda^{q(1+\gamma)} \left(\int_{B(0,b)\cap E} \left(\frac{h(|x|)}{|x|^{n\delta}}\right)^{1+\gamma} dx\right)^{1/p} \left(\int_{(\mathbb{R}^n\setminus B(0,b))\cap E} \left(\frac{h(|x|)}{|x|^{n\delta}}\right)^{1+\gamma} dx\right)^{1/p'}$$
$$\leq \frac{C}{n(\delta(1+\gamma)-1)} \left(\frac{\mu^n}{b^n}\right)^{\frac{\delta-1+\gamma(\delta-p')}{p'}}.$$

Arguing as in [12] we obtain that if $p' \leq \delta$ then $\frac{\delta-1+\gamma(\delta-p')}{p'} \geq 0$ for all $\gamma \in (0,1)$ and if $p' > \delta$, we choose $\gamma \in (0, \frac{\delta-1}{p'-\delta}]$ to guarantee that $\frac{\delta-1+\gamma(\delta-p')}{p'} \geq 0$. So use that $\mu < b$ to ensure that there exists $\gamma \in (0,1)$ such that (3.1) holds.

(2) For $\delta < 1$, the weight $w(x) = \frac{1}{|x|^{n\delta}}$ belongs to A_p, then there exists $\gamma \in (0,1)$ such that $w^{1+\gamma} \in A_p \subset A_p(T^*)$ and it is easy to see that then, for h increasing, $(h(|x|)w(x))^{1+\gamma} \in A_p(T^*)$, therefore the result follows by Proposition 2.7. □

The following proposition deals with one-sided weights. For the definition and basic results of A_p^+ classes and one-sided Hardy–Littlewood maximal operators, see [13].

Proposition 3.2. *Let $p > 1$ and $w : (0,\infty) \to (0,\infty)$ be a weight belonging to A_p^+. Then the weight defined in $\mathbb{R}^n$ by $v(x) = w(|x|)$ belongs to $A_p(T^*)$.*

Proof. This proof follows the ideas of the proof of Lemma 2.4 in [25]. Let $b > 0$. Choose $x_0 = b$ and define an increasing sequence recursively as follows: if x_k was already chosen, then pick $x_{k+1} > x_k$ such that

$$\int_{B(0,x_{k+1})} v = 2\int_{B(0,x_k)} v.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(0,b)} &\frac{v^{1-p'}(x)}{|x|^{np'}}\,dx\left(\int_{B(0,b)} v\right)^{p'}\\ &=\sum_{k=0}^{\infty}\frac{1}{2^{kp'}}\int_{\{x_k\le|x|<x_{k+1}\}}\frac{v^{1-p'}(x)}{|x|^{np'}}\,dx\left(\int_{B(0,x_k)} v\right)^{p'} \qquad (3.2)\\ &=\sum_{k=0}^{\infty}\frac{1}{2^{kp'}}\int_{\{x_k\le|x|<x_{k+1}\}}\left(\frac{1}{|x|^n}\int_{B(0,x_k)} v\right)^{p'} v^{1-p'}(x)\,dx\end{aligned}$$

Observe that $0<r<x_k\le|x|<x_{k+1}$ implies that $\frac{r^{n-1}}{|x|^n}\le\frac{1}{|x|}$. This gives that

$$\begin{aligned}\frac{1}{|x|^n}\int_{B(0,x_k)} v=\frac{C}{|x|^n}\int_0^{x_k} r^{n-1}w(r)\,dr &\le \frac{C}{|x|}\int_0^{|x|} w\chi_{(0,x_{k+1})}(r)\,dr\\ &\le CM^-(w\chi_{(0,x_{k+1})})(|x|),\end{aligned}$$

a.e. x with $x_k\le|x|<x_{k+1}$. Then, using that $w\in A_p^+\Leftrightarrow w^{1-p'}\in A_{p'}^-$ we can continue inequality (3.2) by

$$\le\sum_{k=0}^{\infty}\frac{1}{2^{kp'}}\int_{B(0,x_{k+1})} v(x)\,dx=C\int_{B(0,b)} v\sum_{k=0}^{\infty}\frac{2^{k+1}}{2^{kp'}}=C\int_{B(0,b)} v.$$

Putting this together with (3.2) we get that $v\in A_p(T^*)$. □

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