REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 66, No. 1, 2023, Pages 153–175
Published online: September 21, 2023
https://doi.org/10.33044/revuma.4355

CHARACTERIZATIONS OF LOCAL A_∞ WEIGHTS AND APPLICATIONS TO LOCAL SINGULAR INTEGRALS

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Dedicated to the memory of our dear colleague and friend, Eleonor Harboure. She will always be in our hearts.

Abstract. In a general geometric setting, we prove different characterizations of a local version of Muckenhoupt A_∞ weights. As an application, we obtain conclusions about the relationship between this class and the one-weight boundedness of local singular integrals from L^∞ to BMO.

1. Introduction

As it is well known, a non negative and locally integrable function ω belongs to the A_p class of Muckenhoupt ([10]) if and only if

$$\sup_B \left(\frac{1}{|B|}\int_B \omega\right)\left(\frac{1}{|B|}\int_B \omega^{-\frac{1}{p-1}}\right)^{p-1} < \infty, \quad \text{in the case } 1<p<\infty,$$

$$\sup_B \left(\frac{1}{|B|}\int_B \omega\right)\frac{1}{\inf_B(\omega)} < \infty, \quad \text{for } p=1,$$

and

$$\omega \in A_\infty = \bigcup_{1\le r<\infty} A_r \quad \text{for } p=\infty.$$

Seeing the myriad of articles and books that have been published during the past 30 or 40 years and whose subject is the study of properties or applications of these weights, it is absolutely unnecessary to talk about how important these classes are in the fields of Partial Differential Equations and Harmonic Analysis. The present work is related to some of those articles. The first of them is [4], due to N. Fujii. There, the author introduces some characterizations of A_∞ and uses them to identify a necessary and sufficient condition for a weight ω such that Calderón–Zygmund integrals Tf, with $\frac{f}{\omega} \in L^\infty$, are of ω-weighted bounded mean oscillation. It is important to remark that the characterizations proved in this article were used by Hytönen, Pérez and Rela ([8]) as a basis for obtaining precise estimations of the constants related to the Reverse Hölder inequality and the boundedness of the Hardy–Littlewood maximal function. Another paper this work is related to is [7], due to E. Harboure and the last two authors, where "local" versions of

2020 *Mathematics Subject Classification.* Primary 42B25; Secondary 28C99.

Muckenhoupt weights in a general geometric setting were introduced (even though it should be mentioned that a one dimensional case in $\mathbb{R}$ was considered before by Nowak and Stempak in [11]) and studied in connection with the boundedness of a local maximal operator. In addition, the latter result was applied to get interior Sobolev type estimates for solutions of differential equations associated to the m-Laplacian.

The aim of this work is to obtain a version of the Fujii's results for the local-A_∞ weights and the geometric setting considered in [7]. In order to accurately state our results, we begin with a precise description of the geometric framework.

Let X be a metric space satisfying the weak homogeneity property, that is, each ball $B(x,r)$ cannot contain more than a fix number N of points whose distance from each other is greater than $\frac{r}{2}$. Also, let Ω be an open proper and non empty subset of X such that all the balls included in Ω are connected sets.

In this context, given $0<\beta<1$, we consider the following family of balls

$$\mathcal{F}_\beta = \{B = B(x_B, r_B) : r_B \leq \beta d(x_B, \Omega^c)\},$$

where x_B and r_B are, respectively, the center and the radius of B, and $d(x_B,\Omega^c)$ is the distance from x_B to the complementary set of Ω. Sometimes, we will refer to the balls in $\mathcal{F}_\beta$ as β-local balls.

In addition, let μ be a Borel measure defined on Ω such that $0<\mu(B)<\infty$ and $\mu(B)\leq C_\beta \mu(\frac{1}{2}B)$ for each $B\in\mathcal{F}_\beta$ and every $\beta\in(0,1)$, where θB denotes the ball with the same center and radius θ-times that of B.

Let us note that $\mu(B)$ is finite for all $B\in \bigcup_{0<\beta<1}\mathcal{F}_\beta$ but this fact is not necessarily true for every ball contained in Ω.

Our first result is about the boundedness of β-local singular integrals from a weighted L^∞ space to a β-local BMO. Here, the term "β-local" makes reference to a close relation with the families $\mathcal{F}_\beta$, as we will see in the definitions and notation below.

Given $0<\beta<1$, we will say that T is a β-local singular integral operator if it satisfies

(1.a) T is bounded on $L^2(\Omega, d\mu)$.
(1.b) There is a kernel $K:\Omega\times\Omega\to\mathbb{R}$ such that for any $f\in L^\infty(\Omega,d\mu)$ with compact support

$$Tf(x) = \int_\Omega K(x,y) f(y)\, d\mu(y) \quad \text{a.e. } x\notin \operatorname{supp}(f),$$

and $Tf(x)=0$ for x such that $\operatorname{supp}(f)\cap B(x,\beta d(x,\Omega^c))=\emptyset$.
(1.c) The kernel satisfies
a) $|K(x,y)|\leq \frac{C}{\mu(B(x,d(x,y)))}$,
b) $|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')|\leq \frac{C}{\mu(B(x,d(x,y)))}\left(\frac{d(x,x')}{d(x,y)}\right)^{\xi_0}$
for some $\xi_0>0$ and whenever $2d(x,x')\leq d(x,y)$.

Remark 1.1. *The second assertion in (1.b), which is the meaning of β-locality, can also be written by asking* $\operatorname{supp}(K) \subset \{(x,y) : d(x,y) < \beta d(x,\Omega^c)\}$.

Definition 1.2. *Given $0 < \beta < 1$ and a non negative function ω in $L^1_{loc}(\Omega, d\mu)$, we will say that a function f belongs to BMO^β_ω if the following two conditions are satisfied*

(1.2.a) There exists $C > 0$ such that

$$\frac{1}{\omega(B)} \int_B |f - m_B f| \, d\mu \leq C$$

for every $B \in \mathcal{F}_{\frac{\beta}{6}}$, where $m_B f = \frac{1}{\mu(B)} \int_B f \, d\mu$.

(1.2.b) There exists $C > 0$ such that

$$\frac{1}{\omega(B)} \int_B |f| \, d\mu \leq C$$

for every $B \in \mathcal{F}_\beta - \mathcal{F}_{\frac{\beta}{6}}$.

If $f \in BMO^\beta_\omega$ we will use $[f]_{BMO^\beta_\omega}$ to denote the norm given by infimum of the constants satisfying (1.2.a) and (1.2.b).

Definition 1.3. *Given $0 < \beta < 1$ and $1 < p < \infty$, we will say that a locally integrable and non negative a.e. function ω belongs to the β-local Muckenhoupt class A^β_p if*

$$\sup_{B \in \mathcal{F}_\beta} \left(\frac{1}{\mu(B)} \int_B \omega \, d\mu \right) \left(\frac{1}{\mu(B)} \int_B \omega^{-\frac{1}{p-1}} \, d\mu \right)^{p-1} < \infty.$$

The case $p = \infty$ is defined as $A^\beta_\infty = \bigcup_{1<p<\infty} A^\beta_p$.

Remark 1.4. *It can be easily proved that $\omega \in A^\beta_p$ if and only if $\omega \in A^\alpha_p$ for every $\alpha, \beta \in (0,1)$.*

We will consider a new class of weights

Definition 1.5. *Let $0 < \beta < 1$ and $p > 0$. We say that a weight ω belongs to the class B^β_p if it satisfies*

$$\sup_{B(x,r) \in \mathcal{F}_\beta} \frac{\mu(B(x,r)) r^p}{\omega(B(x,r))} \int_{S_\beta(B(x,r)) - B(x,r)} \frac{\omega(y)}{\mu(B(x, d(x,y))) d(x,y)^p} \, d\mu(y) < \infty,$$

where $S_\beta(B) = \bigcup_{x \in B} B(x, \beta d(x, \Omega^c))$.

Now, we are in position to state our first result.

Theorem 1.6. *Let $0 < \beta < 1$ and T be a β-local singular integral. If $\omega \in A^\beta_\infty \cap B^\beta_{\xi_0}$, where ξ_0 is the exponent appearing in (1.c) associated with T, there exists $C > 0$ such that $[Tf]_{BMO^\beta_\omega} \leq C \|f/\omega\|_\infty$ for every f.*

Our following result is a characterization of the class $A_\infty^\beta(\Omega)$ that involves the maximal operator associated with each family $\mathcal{F}_\beta$ defined as

$$M_\beta f(x) = \sup_{x \in B \in \mathcal{F}_\beta} \frac{1}{\mu(B)} \int_B |f(x)|\, d\mu$$

for $f \in L^1_{loc}(\Omega)$ and $x \in \Omega$.

Theorem 1.7. *Given $0 < \beta < 1$, the following conditions are equivalent*

(1.8.a) $\omega \in A_\infty^\beta$

(1.8.b) There exists $C > 0$ such that

$$\int_{\tilde{B}} M_\beta(\omega \mathcal{X}_B)\, d\mu \le C \int_{\frac{1}{2}B} \omega \, d\mu$$

for every $B \in \mathcal{F}_\beta$, where $\tilde{B} = 5B$ if $5B \in \mathcal{F}_\beta$ and $\tilde{B} = \mathcal{N}_\beta(B) := \bigcup_{\substack{V \in \mathcal{F}_\beta \\ V \cap B \neq \emptyset}} V$ if $5B \notin \mathcal{F}_\beta$.

(1.8.c) There exists $C > 0$ such that

$$\int_B \omega \log^+ \frac{\omega}{m_B \omega}\, d\mu \le C \int_{\frac{1}{2}B} \omega \, d\mu$$

for every $B \in \mathcal{F}_\beta$.

(1.8.d) The weight ω is doubling on $\mathcal{F}_\beta$ and, for each $\varepsilon \in (0,1)$, there exists $\theta \in (0,1)$ such that if $B \in \mathcal{F}_\beta$ and $E \subset B$ satisfy $\mu(E) \le \theta \mu(B)$, then $\omega(E) \le \varepsilon \omega(B)$.

In the particular case $X = \mathbb{R}^n$ with the usual euclidean metric and the Lebesgue measure, Theorem 1.6 has, in a certain sense, a converse. In order to enunciate it we need the Riesz transforms and same local versions. We recall that the j-th Riesz transform, $j = 1, \ldots, n$, of a locally integrable function f is given by

$$R_j f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy.$$

On the other hand, given $0 < \beta < 1$ and a smooth radial cut function η defined on $\mathbb{R}$ such that $0 \le \eta \le 1, \eta(t) = 1$, if $|t| \le \frac{1}{2}$ and $\eta(t) = 0$ when $|t| \ge 1$, we will say that the operator

$$R_j^{\beta,\eta} f(x) = \text{p.v.} \int_\Omega \frac{x_j - y_j}{|x-y|^{n+1}} \eta\left(\frac{|x-y|}{\beta d(x, \Omega^c)}\right) f(y)\, dy$$

is a β-local j-th Riesz type transform.

With these operators we can add a couple of statements to Theorem1.7.

Theorem 1.8. *Given $0 < \beta < 1$, when X is the usual euclidean space $\mathbb{R}^n$ equipped with the Lebesgue measure all the statements in Theorem 1.7 remain equivalent and, in addition, equivalent to each of the following conditions:*

(1.11.a) There exists $C > 0$ such that

$$\int_{\tilde{B}} |R_j(\omega\mathcal{X}_B)|\,dx \le C\int_B \omega\,dx$$

for $j = 1,\dots,n$ and every $B \in \mathcal{F}_\beta$.
(1.11.b) Given η as before, there exists $C > 0$ such that

$$\int_{\tilde{B}} |R_j^{\beta,\eta}(\omega\mathcal{X}_B)|\,dx \le C\int_B \omega\,dx$$

for $j = 1,\dots,n$ and every $B \in \mathcal{F}_\beta$. In both statements $\tilde{B}$ is defined as in (1.8.b).

The latter Theorem allows us to get, as we said before, a certain kind of converse of Theorem 1.6

Theorem 1.9. *Let $0 < \beta < 1$ and let $R_j^{\beta,\eta}, j = 1,\dots,n$ a β-local Riesz type transforms. If there exists $C > 0$ such that $\left[R_j^{\beta,\eta} f\right]_{BMO_\omega^\beta} \le C\|\frac{f}{\omega}\|_\infty$ for every f satisfying $\frac{f}{\omega} \in L^\infty(\Omega)$ and every $j, j = 1,\dots,n$, then ω belongs to $A_\infty^\beta(\Omega) \cap B_1^\beta$.*

The structure of the paper is as follows: Section 2 is devoted to the proof of Theorem 1.6. Section 3 contains the proof of Theorem 1.8 and a local version of the Calderón–Zygmund decomposition interesting by itself. Section 4 focuses on the proof of Theorem 1.11. Finally, Section 5 contains the proofs of Theorem 1.12 and some properties of the classes B_p^β.

2. Proof of Theorem 1.6

The first step in proving Theorem 1.6 is to ensure that a β-local singular integral T is well defined on functions f such that $\frac{f}{\omega} \in L^\infty(\Omega, d\mu)$ for a weight ω in A_∞^β. Towards this aim, we start recalling that, given $a \in (0, \frac{\beta}{80})$, Ω can be covered by countable family of balls $\{B(x_i, r_i)\}_i$ having finite overlapping and such that $\frac{a}{2}d(x_i, \Omega^c) \le r_i \le ad(x_i, \Omega^c)$ (see Lemma 2.3 and Remark 2.4 in [7]). Now let f such that $\frac{f}{\omega} \in L^\infty(\Omega, d\mu)$. For $x \in B_i = B(x_i, r_i)$, from Remark 1.1, we can write

$$Tf(x) = T(f\mathcal{X}_{S_\beta(B_i)})(x), \tag{2.1}$$

where $S_\beta(B_i)$ is as in definition(1.5). Notice that T is bounded from $L^p(\Omega, d\mu)$ to $L^p(\Omega, d\mu)$ for $1 < p < \infty$ (Theorem 4.1 in [7]). Then, all we have to do in order to prove that Tf is finite a.e. on B_i and, hence on Ω, is to prove that $f\mathcal{X}_{S_\beta(B_i)} \in L^p(\Omega, d\mu)$ for some $p > 1$. This, in turn will follow from the fact that $|f| \le ||\frac{f}{\omega}||_\infty\omega$ once we prove ω has better integrability than $L^1_{\text{loc}}(\Omega, d\mu)$. In order to prove it, we need to look at the theory of Muckenhoupt weights in spaces of homogeneous type. With this in mind, we start recalling that a space of homogeneous type is a non empty set Y equipped with a quasi distance τ and a doubling Borel measure υ. By a quasi distance we mean a function $\tau : Y \times Y \to \mathbb{R}_0^+$ satisfying the following properties:

(2.a) $\tau(x,y) = 0 \Leftrightarrow x = y$.
(2.b) $\tau(x,y) = \tau(y,x)$ for every $x, y \in Y$.
(2.c) $\tau(x,y) \leq K(\tau(x,z) + \tau(z,y))$ for every $x, y, z \in Y$ and a certain constant K.

In addition, we say that a measure υ is doubling if $\upsilon(2B) \leq C\upsilon(B)$ for every ball B in Y. In the usual euclidean space $\mathbb{R}^n$ with the Lebesgue measure it is immediate to see that each ball B is, in turn, a space of homogeneous type in itself. This is not necessarily true in any space of homogeneous type. However, Macías and Segovia, in [9], proved that, for any space (Y, τ, υ) as before, there exists a metric δ satisfying $\delta \leq \tau \leq 3\delta$ such that every δ-ball is a space of homogeneous type with the measure υ. This remarkable achievement allowed them to prove that a well-known result due to Coifman and Fefferman ([3]) is still valid for A_p weights in spaces of homogeneous type (i.e., weights satisfying inequality in Definition (1.3) but for every ball in the space). Their extension can be stated as follows.

Theorem 2.2. *([9]) Let (Y, τ, υ) be a space of homogeneous type. If ω satisfies condition A_p, then there exists $\varepsilon > 0$ and $C > 0$ such that*

$$\frac{1}{\upsilon(B)} \int_B \omega^{1+\varepsilon}\, d\upsilon \leq C \left(\frac{1}{\upsilon(B)} \int_B \omega\, d\upsilon \right)^{1+\varepsilon}$$

holds for every ball B.

On the other hand, in [6], Harboure, Viviani and the second author proved that in our present geometrical setting the construction devised by Macías and Segovia can still be carried out to get a metric δ equivalent to the d given. Moreover, they proved that each δ-ball Q such that $4Q \subset \Omega$ is a space of homogeneous type with δ and the restriction to Q of the measure μ. It was also proved in [7] that a weight ω in $A_p^\beta(\Omega)$ for $\beta \in (0,1)$ given, is a weight in the class A_p defined on the space Q for every δ-balls Q with $4Q \subset \Omega$. Then, Theorem 2.2 allows us to get inequality 2.2 with $B = Q$ and $\upsilon = \mu$ for every δ-balls Q as before.

A careful monitoring of the constants involved shows that they only depend on β. It is an easy consequence of the relation between δ and d that the same result holds for every d-ball $B \in \mathcal{F}_\beta$ whenever $\beta < \frac{1}{3}$.

In this way, for $\beta < \frac{1}{3}$, we see that weights in $A_\infty^\beta(\Omega)$ possesses a better integrability on balls belonging to $\mathcal{F}_\beta$. At this point we have two problems: the restriction on β and the fact that the set in (2.1) is not a ball. The latter can be overcome applying the following lemma.

Lemma 2.3. *Let $\varepsilon_0 \in (0,1]$. Then, given $\lambda \in (0, \varepsilon_0)$ we have $\lambda + \varepsilon_2 \in (0, \varepsilon_0)$ and $S_\lambda(B) \subset B(x_0, (\lambda + \varepsilon_2) d(x_0, \Omega^c))$ for every ball $B = B(x_0, r) \in \mathcal{F}_{\varepsilon_1 \lambda}$, with $0 < \varepsilon_1 < \frac{1}{2} \min\left(1, \frac{\varepsilon_0 - \lambda}{\lambda^2 + \lambda}\right)$ and $\varepsilon_2 = \varepsilon_1(\lambda^2 + \lambda)$.*

Proof. Let $B = B(x_0, r) \in \mathcal{F}_{\varepsilon_1\lambda}$, with ε_1 to be chosen later. Then, for $x \in B$ and $z \in B(x, \lambda d(x, \Omega^c))$ we have

$$\begin{aligned} d(x_0, z) &\le d(x_0, x) + d(x, z) \\ &< \varepsilon_1 \lambda d(x_0, \Omega^c) + \lambda d(x, \Omega^c) \\ &< \varepsilon_1 \lambda d(x_0, \Omega^c) + \lambda(1 + \varepsilon_1\lambda) d(x_0, \Omega^c) \\ &= (\lambda + \varepsilon_1\lambda + \varepsilon_1\lambda^2) d(x_0, \Omega^c). \end{aligned}$$

Now, choosing $\varepsilon_2 = \varepsilon_1(\lambda + \lambda^2)$ with $0 < \varepsilon_1 < \frac{1}{2}\min\left(1, \frac{\varepsilon_0 - \lambda}{\lambda^2 + \lambda}\right)$, from the last inequality we easily obtain $S_\lambda(B) \subset B(x_0(\lambda + \varepsilon_2) d(x_0, \Omega^c))$. Furthermore, it is clear that $0 < \lambda + \varepsilon_2 < \varepsilon_0$. □

So, taking $\varepsilon_0 = \frac{1}{3}$ and $\lambda = \beta$ in the lemma above and applying it for each $S_\beta(B_i)$ in (2.1), we get that there exists $p > 1$ such that $f\mathcal{X}_{S_\beta(B_i)} \in L^p(\Omega, d\mu)$ for every i. Consequently, Tf is finite a.e. in Ω, as soon as T is a β-local singular integral with $\beta < \frac{1}{3}$. This latter restriction can be avoided by using a technique applied in [5]. If T is a β-local operator with $1 > \beta \ge \frac{1}{3}$, we take a smooth radial cut function as in (1.10) and define

$$T_0 f(x) = \int_\Omega K(x, y)\left(1 - \eta\left(\frac{d(x, y)}{\alpha d(x, \Omega^c)}\right)\right) f(y)\, d\mu(y),$$

with $0 < 2\alpha < \frac{1}{3}$ and K the kernel associated with T. It is very easy to see that $T_0 f(x)$ is finite for every $x \in \Omega$ and every f such $\frac{f}{\omega} \in L^\infty(\Omega, d\mu)$. Additionally, it can be proved, as in [7], that $T - T_0$ is a 2α-local singular integral. Then, as we saw earlier, it is well defined as well, and so it is T.

Now, we can proceed with the rest of the proof.

Proof of Theorem 1.6 Let $\omega \in A_\infty^\beta \cap B_{\xi_0}^\beta$. Then, we know that ω belongs to A_p^β for some $p > 1$. Given f such that $\frac{f}{\omega} \in L^\infty(\Omega, d\mu)$ and $B_0 = B(x_0, r) \in \mathcal{F}_{\frac{\beta}{6}}$, we split f as $g + h$, where $g = f\mathcal{X}_{2B_0}$. Now we can write

$$\begin{aligned} \frac{1}{\omega(B_0)} \int_{B_0} |Tf - m_{B_0} Tf|\, d\mu &\le \frac{1}{\omega(B_0)} \int_{B_0} |Tg - m_{B_0} Tg|\, d\mu \\ &\quad + \frac{1}{\omega(B_0)} \int_{B_0} |Th - m_{B_0} Th|\, d\mu \\ &= I + II. \end{aligned} \tag{2.4}$$

From the reasoning done at the beginning of this section we know that g belongs to $L^q(\Omega, d\mu)$ for $q = 1 + \varepsilon$ where ε is the exponent given by Theorem 2.2 applied in our context. Then, since T is bounded from $L^q(\Omega, d\mu)$ to $L^q(\Omega, d\mu)$ (see [5]), Hölder's inequality allows us to get

$$
\begin{aligned}
I &\leq \frac{2}{\omega(B_0)} \int_{B_0} |Tg| \, d\mu && (2.5)\\
&\leq \frac{2\mu(B_0)^{1-\frac{1}{q}}}{\omega(B_0)} \left(\int_{B_0} |Tg|^q \, d\mu \right)^{\frac{1}{q}}\\
&\leq C \frac{2\mu(B_0)^{1-\frac{1}{q}}}{\omega(B_0)} \left(\int_{2B_0} |f|^q \, d\mu \right)^{\frac{1}{q}}\\
&\leq C \left\| \frac{f}{\omega} \right\|_\infty \frac{2\mu(B_0)}{\omega(B_0)} \left(\frac{1}{\mu(B_0)} \int_{2B_0} \omega^q \, d\mu \right)^{\frac{1}{q}}\\
&\leq C \left\| \frac{f}{\omega} \right\|_\infty ,
\end{aligned}
$$

where the last inequality follows from Theorem 2.2, which, as it is clear from the discussion above this proof, holds for $2B_0, \omega$ and μ.

In regard to II, applying (1.c), we can estimate it in the following way:

$$
\begin{aligned}
II &\leq \frac{1}{\omega(B_0)\mu(B_0)} \int_{B_0} \int_{B_0} |Th(x) - Th(y)| \, d\mu(x) d\mu(y) && (2.6)\\
&\leq \frac{1}{\omega(B_0)\mu(B_0)} \int_{B_0} \int_{B_0} \int_{S_\beta(2B_0)-2B_0} |K(x,z) - K(y,z)||f(z)| \, d\mu(z) d\mu(x) d\mu(y)\\
&\leq \frac{C}{\omega(B_0)\mu(B_0)} \int_{B_0} \int_{B_0} \int_{S_\beta(2B_0)-2B_0} \frac{d(x,x_0)^{\xi_0} + d(y,x_0)^{\xi_0}}{\mu(B(x_0, d(x_0,z))) d(x_0,z)^{\xi_0}}\\
&\qquad\qquad \times |f(z)| \, d\mu(z) d\mu(x) d\mu(y)\\
&\leq C \frac{\mu(B_0) r^{\xi_0}}{\omega(B_0)} \int_{S_\beta(2B_0)-2B_0} \frac{|f(z)|}{\mu(B(x_0, d(x_0,z))) d(x_0,z)^{\xi_0}} \, d\mu(z)\\
&\leq \left\| \frac{f}{\omega} \right\|_\infty C \frac{\mu(B_0) r^{\xi_0}}{\omega(B_0)} \int_{S_\beta(2B_0)-2B_0} \frac{\omega(z)}{\mu(B(x_0, d(x_0,z))) d(x_0,z)^{\xi_0}} \, d\mu(z)\\
&\leq C \left\| \frac{f}{\omega} \right\|_\infty ,
\end{aligned}
$$

where the last inequality follows from the hypothesis on ω.

So, altogether (2.4), (2.5) and (2.6) give us all the information we need about the behaviour of the oscillations of Tf on the balls in $\mathcal{F}_{\frac{\beta}{6}}$.

Now, let us take care of the averages on balls belonging to $\mathcal{F}_\beta - \mathcal{F}_{\frac{\beta}{6}}$. Let $B_0 = B(x_0, r)$ be one of them. Then, reasoning as before, we obtain

$$
\begin{aligned}
\frac{1}{\omega(B_0)} \int_{B_0} |Tf| \, d\mu &= \frac{1}{\omega(B_0)} \int_{B_0} |T(f\mathcal{X}_{S_\beta(B_0)})| \, d\mu\\
&\leq \frac{\mu(B_0)}{\omega(B_0)} \left(\frac{1}{\mu(B_0)} \int_{B_0} |T(f\mathcal{X}_{S_\beta(B_0)})|^q \, d\mu \right)^{\frac{1}{q}} ,
\end{aligned}
$$

where, once again, $q = 1+\varepsilon$ with ε given by Theorem 2.2. Theorem 41 in [7] lead us to

$$\begin{aligned}\frac{1}{\omega(B_0)}\int_{B_0}|Tf|\,d\mu &\le C\frac{\mu(B_0)}{\omega(B_0)}\left(\frac{1}{\mu(B_0)}\int_{S_\beta(B_0)}|f|^q\,d\mu\right)^{\frac{1}{q}}\\ &\le C\left\|\frac{f}{\omega}\right\|_\infty\frac{\mu(B_0)}{\omega(B_0)}\left(\frac{1}{\mu(B_0)}\int_{S_\beta(B_0)}\omega^q\,d\mu\right)^{\frac{1}{q}}.\end{aligned}$$

It is obvious that $S_\beta(B_0) \subset \bigcup_{\substack{B\in\mathcal{F}_{\beta'}\\ B\cap B_0\neq\emptyset}} B$, for any $\beta' \in (\beta, 1)$.

Denoting the latter set by $\mathcal{N}_{\beta'}(B_0)$, as in [7], Lemmas 2.3 and 3.1 there allow us to get a finite number M of balls $B_i = B(x_i, r_i)$ such that $\mathcal{N}_{\beta'}(B_0) \subset \bigcup_i B_i$, $\mu(B_i) \simeq \mu(B_0)$, $\omega(B_i) \simeq \omega(B_0)$, $\frac{a}{2}d(x_i,\Omega^c) \le r_i \le a d(x_i,\Omega^c)$ for some fixed $a < \frac{\beta'}{80}$, $i = 1,\dots,M$, with M only depending on β and β'. Consequently, if we chose β' close enough to β, Theorem 2.2 can be applied for each B_i to get

$$\begin{aligned}\frac{1}{\omega(B_0)}\int_{B_0}|Tf|\,d\mu &\le C\left\|\frac{f}{\omega}\right\|_\infty\frac{\mu(B_0)}{\omega(B_0)}\frac{1}{\mu(B_0)}\int_{B_0}\omega\,d\mu\\ &= C\left\|\frac{f}{\omega}\right\|_\infty,\end{aligned}$$

which completes the proof of our Theorem.

3. Proof of Theorem 1.7

The proof of Theorem 1.7 requires a local version of the Calderón–Zygmund decomposition. Our proof of it is based on techniques developed by H. Aimar and R. Macías in the setting of spaces of homogeneous type ([1], [2]).

Lemma 3.1. *Let $0 < \beta < 1$. Given $B \in \mathcal{F}_\beta$ and a non negative function $f \in L^1_{loc}(\Omega, d\mu)$, with $\operatorname{supp} f \subset B$, for each $\lambda \ge m_B f$ there exists a sequence $\{B_j\}$ of disjoint balls in $\mathcal{F}_\beta$ such that*

(3.1.a) $m_{\tilde{B}_j} f \le \lambda < m_{B_j} f$, for every j;

(3.1.b) $m_V f \le \lambda$, for every $V \in \mathcal{F}_\beta$ whose center belongs to $\Omega - \bigcup_j \tilde{B}_j$;

where $\tilde{B}_j$ is defined as in (1.8.b).

Proof. Let us assume $E = \{y \in \Omega; M_\beta f(y) > \lambda\} \neq \emptyset$. If $E = \emptyset$, then (3.1.b) holds for every $V \in \mathcal{F}_\beta$ with center in Ω and the lemma follows. Clearly, if $\Gamma = \{V \in \mathcal{F}_\beta / m_V f > \lambda\}$, we get $E = \bigcup_{V\in\Gamma} V$. Now, it is immediate to see that $V \cap B \neq \emptyset$ for every $V \in \Gamma$, which implies $B \subset \mathcal{N}_\beta(V)$ and $V \subset \mathcal{N}_\beta(B)$. So, we have

$$\frac{1}{\mu(\mathcal{N}_\beta(V))}\int_{\mathcal{N}_\beta(V)} f\,d\mu = \frac{1}{\mu(\mathcal{N}_\beta(V))}\int_{\mathcal{N}_\beta(V)\cap B} f\,d\mu \le \lambda \tag{3.2}$$

for every $V \in \Gamma$. Taking $V = B(x,r) \in \Gamma$, we define

$$\gamma_x = \sup\{t \in (0, \beta d(x,\Omega^c))/m_{B(x,t)}f > \lambda\},$$

which satisfies $r \leq \gamma_x \leq \beta d(x,\Omega^c)$. If $r < \gamma_x$, we take δ in $(0, \frac{4}{5}\gamma_x)$, and t_x in $(\max(r, \gamma_x - \delta), \gamma_x]$ such that $m_{B(x,t_x)}f > \lambda$. If $5t_x \leq \beta d(x,\Omega^c)$, since $5t_x > 5(\gamma_x - \delta) > \gamma_x$, we have $m_{B(x,5t_x)}f \leq \lambda$. Then, taking this and (3.2) into account, we have

$$m_{\tilde{B}(x,t_x)}f \leq \lambda < m_{B(x,t_x)}f.$$

On the other hand, if $r = \gamma_x$, we get the above inequality by choosing $t_x = \gamma_x$. Proceeding in this way for each $x \in A := \{y/y \text{ is center of a ball in } \Gamma\}$ we obviously obtain a covering of E by the sets $\tilde{B}_x$ where $B_x = B(x,t_x)$. Note that these balls are in $\mathcal{N}_\beta(B)$. Then, from Lemma 2.3 in [6], it follows that their radii are uniformly bounded. So, Lemma (1.11.a) in that paper (local Vitali) allows us to get a numerable disjoint subfamily of $\{B_x\}_{x\in A}$, say $\{B_j\}$ such that $E \subset \bigcup_j \tilde{B}_j$. This is the sequence we were looking for. □

With this lemma we are in position to prove Theorem 1.7.

Proof of Theorem 1.7. Suppose (1.8.a) holds, that is $\omega \in A^\beta_\infty$. Then, by definition, we get $\omega \in A^\beta_p$ for some $p \in (1,\infty)$. Let $B = B(x_0, r) \in \mathcal{F}_\beta$. If $r < \frac{\beta}{3} d(x_0,\Omega^c)$, from the discussion preceding Lemma 2.3, we know ω satisfies a reverse Hölder's inequality on B for some exponent $q > 1$. In case $\frac{\beta}{3} d(x_0,\Omega^c) \leq r < \beta d(x_0,\Omega^c)$, from Lemma 2.3 in [7], we can cover B with the union of a fixed number M, not depending on B, of balls belonging to $\mathcal{F}_{\frac{\beta}{3}}$ and having finite over lapping. Moreover, the union of such balls and B have comparable measures. It follows that ω satisfy a reverse Hölder's inequality on B with the same exponent q. With this in mind, taking into account that M_β is bounded on $L^q(\Omega, d\mu)$ (Theorem 1.1 in [6]), Hölder's inequality allows us to get

$$\begin{aligned}
\int_{\tilde{B}} M_\beta(\omega\mathcal{X}_B)\, d\mu &\leq \left(\int_\Omega M_\beta(\omega\mathcal{X}_B)^q\, d\mu\right)^{\frac{1}{q}} \mu(\tilde{B})^{1-\frac{1}{q}} \\
&\leq C \left(\int_B \omega^q\, d\mu\right)^{\frac{1}{q}} \mu(B)^{1-\frac{1}{q}} \\
&\leq C \int_B \omega\, d\mu,
\end{aligned}$$

which, taking into account that ω is doubling on $\mathcal{F}_\beta$, lead us to (1.8.b).

Now assume (1.8.b) holds. Let $B \in \mathcal{F}_\beta$. If $5B \notin \mathcal{F}_\beta$ from our hypothesis it follows

$$\begin{aligned}
\int_{\{M_\beta f > m_B\omega\}} M_\beta f\, d\mu &= \int_{\mathcal{N}_\beta(B)\cap\{M_\beta f > m_B\omega\}} M_\beta f\, d\mu \qquad (3.3)\\
&\leq C \int_{\frac{1}{2}B} \omega\, d\mu
\end{aligned}$$

for $f = \omega\mathcal{X}_B$. On the other hand, if $5B \in \mathcal{F}_\beta$, (1.8.b) lead us to

$$\int_{\{M_\beta f > m_B\omega\}} M_\beta f \, d\mu \leq C \int_{\frac{1}{2}B} \omega \, d\mu + \int_{(\mathcal{N}_\beta(B)-5B)\cap\{M_\beta f > m_B\omega\}} M_\beta f \, d\mu. \tag{3.4}$$

It is not difficult to see that $B \subset 2V$ for every ball $V \in \mathcal{F}_\beta(\Omega)$ such that $V \cap B \neq \emptyset$ and $V \cap (\mathcal{N}_\beta(B) - 5B) \neq \emptyset$. Then $B \subset \tilde{V}$ and so $\mu(B) \leq C\mu(V)$, which, in turn, implies $M_\beta f(y) \leq C m_B \omega$ for every $y \in \mathcal{N}_\beta(B) - 5B$. In consequence, from (3.4), the weak type boundedness $(1,1)$ of M_β (Theorem 1.1 in [6]), and the fact that ω is doubling on $\mathcal{F}_\beta$ (it is obvious from (1.8.b)), we get

$$\begin{aligned}\int_{\{M_\beta f > m_B\omega\}} M_\beta f \, d\mu &\leq C \left(\int_{\frac{1}{2}B} \omega \, d\mu + m_B\omega\mu(\{M_\beta f > m_B\omega\}) \right) \\ &\leq C \int_{\frac{1}{2}B} \omega \, d\mu.\end{aligned} \tag{3.5}$$

Besides that, taking $\lambda > m_B\omega$, Lemma 3.1 gives us a sequence $\{B_j\}$ of disjoint balls in $\mathcal{F}_\beta$ such that

$$\bigcup_j B_j \subset \{M_\beta f > \lambda\} \subset \bigcup_j \tilde{B}_j.$$

Applying (3.1.a) we can obtain

$$\begin{aligned}\mu\left(\{M_\beta f > \lambda\}\right) &\geq \sum_j \mu(B_j) \\ &\geq \frac{C}{\lambda} \int_{\bigcup_j \tilde{B}_j} f \, d\mu \\ &\geq \frac{C}{\lambda} \int_{\{M_\beta f > \lambda\}} f \, d\mu \\ &\geq \frac{C}{\lambda} \int_{\{f > \lambda\}} f \, d\mu.\end{aligned}$$

Integrating both sides with respect to λ, Fubini's Theorem together with (3.3) and (3.5) lead us to

$$\begin{aligned}\int_{m_B\omega}^{\infty} \frac{1}{\lambda} \left(\int_{\{f>\lambda\}} f \, d\mu \right) d\lambda &\leq C \int_{m_B\omega}^{\infty} \mu\left(\{M_\beta f > \lambda\}\right) d\lambda \\ &\leq C \int_{\{M_\beta f > m_B\omega\}} M_\beta f \, d\mu \\ &\leq C \int_{\frac{1}{2}B} \omega \, d\mu.\end{aligned} \tag{3.6}$$

Recalling that $f = \omega\mathcal{X}_B$, Fubini's Theorem again allows us to get

$$\begin{aligned}\int_{m_B\omega}^{\infty} \frac{1}{\lambda}\int_{\{f>\lambda\}} f\,d\mu\,d\lambda &= \int_{\{f>m_B\omega\}} f\log\left(\frac{f}{m_B\omega}\right)d\mu\\ &= \int_B \omega\log^+\left(\frac{\omega}{m_B\omega}\right)d\mu,\end{aligned}$$

which, together with (3.6), obviously proves (1.8.c).

Let $\varepsilon \in (0,1)$. Given $B \in \mathcal{F}_\beta$ and $E \subset B$ such that $\mu(E) > 0$, we define $E_0 = \{x \in E/\omega(x) > \frac{\varepsilon}{2\mu(E)}\int_B \omega\,d\mu\}$. Then, assuming (1.8.c) holds, we obtain

$$\begin{aligned}\log^+\left(\frac{\varepsilon\mu(B)}{2\mu(E)}\right)\int_{E_0}\omega\,d\mu &\le \int_B \omega\log^+\left(\frac{\omega}{m_B\omega}\right)d\mu\\ &\le C\int_B \omega\,d\mu.\end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}\int_{E-E_0}\omega\,d\mu &\le \varepsilon\frac{\mu(E-E_0)}{2\mu(E)}\int_B \omega\,d\mu\\ &\le \frac{\varepsilon}{2}\int_B \omega\,d\mu.\end{aligned}$$

This inequality assures us that if $\omega(E) > \varepsilon\omega(B)$, then $\omega(E_0) > \frac{\varepsilon}{2}\omega(B)$, and so, from (3.7)

$$\log^+\frac{\varepsilon\mu(B)}{2\mu(E)} \le C\frac{\omega(B)}{\omega(E_0)} < \frac{2C}{\varepsilon}.$$

Consequently, $\frac{\varepsilon}{2}e^{-\frac{2C}{\varepsilon}}\mu(B) < \mu(E)$. Finally, taking $\theta = \frac{\varepsilon}{2}e^{-\frac{2C}{\varepsilon}}$, which belongs to $(0,1)$, and noting that (1.8.c) implies ω is doubling on $\mathcal{F}_\beta$, we prove (1.8.d).

In order to see that (1.8.d) implies (1.8.a), we consider again, as in the beginning of section 2, the metric δ such that $\delta \le d \le 3\delta$ and each δ-ball $B_\delta(x_0,r)$, with $r < (\beta/3)d(x_0,\Omega^c)$, is a space of homogeneous type endowed with the restriction of μ. Given one of these balls, say B_δ, we have $B_\delta \subset 3B_d$, where B_d denotes the d-ball with same centre and radius as B_δ. Then, since $B_d \in \mathcal{F}_{\frac{\beta}{3}}$, we get

$$\omega(3B_d) \le \mathcal{D}\omega(B_d) \le \mathcal{D}\omega(B_\delta),$$

where $\mathcal{D}$ denotes the doubling constant of ω associated to $\mathcal{F}_\beta$.

Let us prove that (1.8.d) holds for δ-balls as well. To this aim we take a δ-ball B_δ as before and consider $E \subset B_\delta$. Then $E \subset B_d$, where B_d denotes, as before, the d-ball with same centre and radios as B_δ. Then, choosing $\varepsilon \in (0,1)$ such that $\varepsilon\mathcal{D} \in (0,1)$, we get $\theta \in (0,1)$ such that

$$\frac{\omega(E)}{\omega(B_\delta)} \le \frac{\omega(E)\mathcal{D}}{\omega(3B_d)} < \mathcal{D}\varepsilon,$$

whenever $\mu(E) < \theta\mu(B_\delta)$, since $\mu(B_\delta) \le \mu(3B_d)$. This proves (1.8.d) for these δ-balls.

A careful examination of the proof of Theorem 2.2 in [9](Theorem (2.2) there) reveals that it can be done by assuming (1.8.d) instead of the A_p condition. Then, there exists $q>1$ such that the inequality

$$\left(\frac{1}{\mu(B)}\int_B \omega^q\,d\mu\right)^{\frac{1}{q}} \le C\frac{1}{\mu(B)}\int_B \omega\,d\mu,$$

holds for every ball $B\in\mathcal{F}_{\frac{\beta}{3}}$. By denoting $dv=\omega d\mu$, the above inequality can be written as follows:

$$\frac{1}{\int_B \omega^{-1}\,dv}\int_B \omega^{q-1}\,dv \le \left(C\frac{1}{\int_B \omega^{-1}\,dv}\int_B \omega\,d\mu\right)^q.$$

It follows easily that $\omega^{-1}\in A^{\frac{\beta}{3}}_{1+\frac{1}{q-1}}$ but with respect to the measure v instead of μ. Then, since (1.8.d) imply ω is doubling on $\mathcal{F}_\beta$, Theorem 2.2 can be applied again to ω^{-1} with the measure v to obtain a reverse Hölder inequality for ω^{-1} respect to v. Reasoning in a similar way as before we get $\omega\in A_p^{\frac{\beta}{3}}$ with respect to μ. Finally, from Remark 1.4 it follows $\omega\in A_\infty^\beta$, as we wanted to prove. $\square$

4. Proof of Theorem 1.8

Here we are in the particular case $X=\mathbb{R}^n$, d the euclidean metric and μ the Lebesgue measure. Let us start proving the following proposition.

Proposition 4.1. *Let ω be a non negative function in $L^1_{loc}(\Omega,d\mu)$ and $\beta\in(0,1)$. The following statements are equivalent.*

(4.1.a) There exists $C>0$ such that

$$\int_{\tilde{B}} |R_j^{\beta,\eta}(\omega\mathcal{X}_B)|\,dx \le C\int_B \omega\,dx,$$

for $j=1,\dots,n$ and every $B\in\mathcal{F}_\beta$.

(4.1.b) There exists $C>0$ such that

$$\int_{\tilde{B}} |R_j(\omega\mathcal{X}_B)|\,dx \le C\int_B \omega\,dx,$$

for $j=1,\dots,n$ and every $B\in\mathcal{F}_\beta$.

Proof. Given $B=B(x_0,r)\in\mathcal{F}_\beta$, notice that $R_j^{\beta,\eta}(\omega\mathcal{X}_B)$ and $R_j(\omega\mathcal{X}_B)$ are finite a.e. in Ω since the operators are of weak type $(1,1)$ (in particular, for $R_j^{\beta,\eta}$, this result was proved in [7], Theorem 4.1). Let us see that (4.1.a) implies (4.1.b). For each j is clear that

$$\begin{aligned}|R_j(\omega\mathcal{X}_B)(x)| &\le |R_j(\omega\mathcal{X}_{B\cap B(x,\frac{\beta}{2}d(x,\Omega^c))})(x)| \\ &\quad + |R_j(\omega\mathcal{X}_{B\cap B^c(x,\frac{\beta}{2}d(x,\Omega^c))})(x)| = I+II\end{aligned}$$

for almost every $x\in\Omega$. We can estimate I as follows:

$$
\begin{aligned}
I &= \left| R_j^{\beta,\eta}(\omega\mathcal{X}_B)(x) - \int_{B^c(x,\frac{\beta}{2}d(x,\Omega^c))} \frac{x_j-y_j}{|x-y|^{n+1}}\eta\left(\frac{|x-y|}{\beta d(x,\Omega^c)}\right)\omega\mathcal{X}_B\,dy\right| \\
&\le |R_j^{\beta,\eta}(\omega\mathcal{X}_B)(x)| + \left(\frac{2}{\beta d(x,\Omega^c)}\right)^n \int_B \omega\,dy.
\end{aligned}
$$

On the other hand, for $x \in \tilde{B}$ we have

$$
II \le \left(\frac{2}{\beta d(x,\Omega^c)}\right)^n \int_B \omega\,dy.
$$

Then, from the estimates of I and II, we get

$$
\begin{aligned}
\int_{\tilde{B}} |R_j(\omega\mathcal{X}_B)|\,dx \le \int_{\tilde{B}} &|R_j^{\beta,\eta}(\omega\mathcal{X}_B)|\,dx \\
&+ 2\left(\frac{2}{\beta}\right)^n \left(\int_{\tilde{B}} \frac{dx}{d(x,\Omega^c)^n}\right)\left(\int_B \omega\,dx\right).
\end{aligned}
$$

If $5r \le \beta d(x_0,\Omega^c)$, it follows that $5r(\frac{1}{\beta}-1) \le d(x,\Omega^c)$ for every $x \in 5B$. Then

$$
\int_{\tilde{B}} \frac{dx}{d(x,\Omega^c)^n} \le \frac{C}{r^n}\int_{5B} dx = C. \tag{4.2}
$$

In the case $5r > \beta d(x_0,\Omega^c)$, we have $\tilde{B} = \mathcal{N}_\beta(B)$. Following the proof of Lemma 2.3 in [6] (see p. 616), we know that there exists a constant C, independent of B, such that $d(x,\Omega^c) \ge Cr$ for every $x \in \mathcal{N}_\beta(B)$. In consequence, we can obtain (4.2) again. Finally, (4.1.a) implies (4.1.b). Taking into account that

$$
R_j^{\beta,\eta}(\omega\mathcal{X}_B)(x) = R_j(\omega\mathcal{X}_B)(x) - \int_B \frac{x_j-y_j}{|x-y|^{n+1}}\left(1-\eta\left(\frac{|x-y|}{\beta d(x,\Omega^c)}\right)\right)\omega\,dy
$$

for almost every $x \in \Omega$, a similar reasoning as before allows us to get that (4.1.b) implies (4.1.a). □

Proof of Theorem 1.8 Proposition 4.1 proves that (4.1.a) and (4.1.b) are equivalent. Let us see that (4.1.b) implies (1.8.b). With this aim in mind, we take a ball $B = B(x_0,r) \in \mathcal{F}_\beta$ and denote $f(x) = \omega(x)\mathcal{X}_B(x)$ and $g(x) = -f(x+y_0)$ with $y_0 \in \mathbb{R}^n$.

Claim: It is possible to choose $y_0 \in \mathbb{R}^n$ such that $|y_0| = \delta r$ with $0 < \delta < \frac{1-\beta}{2}$, and

$$
\int_{\mathbb{R}^n} |R_j(f+g)|\,dx \le C\int_{\mathbb{R}^n} f\,dx, \tag{4.3}
$$

with C a constant not depending on B.

Assuming the claim is valid and proceeding in an analogous way as in the proof of (ii)⇒(iii) in Theorem 1 of [4], we obtain

$$
Mf(x) \le C\left(|B|^{-1}\int f\,dx + \sup_{t>0}|((f+g)*P_t)(x)|\right)
$$

for every $x \in \tilde{B}$, where M denotes the classical Hardy–Littlewood maximal and P_t is the Poisson kernel. Then, integrating both sides over $\tilde{B}$, we get

$$\begin{aligned}\int_{\tilde{B}} M_\beta f\,dx &\le \int_{\tilde{B}} Mf\,dx\\ &\le C\left(\int_{\tilde{B}} f\,dx + \int_{\tilde{B}} \sup_{t>0} |(f+g)*P_t|\,dx\right).\end{aligned}$$

As in [4] it can be proved that the second integral on the right side is bounded by a constant times the first. This allows us to obtain

$$\int_{\tilde{B}} M_\beta(\omega\mathcal{X}_B)\,dx \le C\int_B \omega\,dx.$$

A careful examination of the proof that (1.8.b) imply (1.8.c) (see Theorem (1.8)) leads to the conclusion that considering the inequality above instead of the one in (1.8.b) allows us to get the inequality in (1.8.c) but with $\omega(B)$ instead of $\omega(\frac{1}{2}B)$ on the right hand side. This, in turn, lead us to (1.8.d) but without assuring the doubling condition. However, in the particular case of μ being the Lebesgue measure in $\mathbb{R}^n$, this is enough to prove ω is doubling on $\mathcal{F}_\beta$. In fact, taking $\varepsilon = \frac{1}{2}$ its corresponding θ in $(0,1)$ and $t = (1-\theta)^{-1/n}$, we get

$$\begin{aligned}\mu(B - t^{-1}B) &= (1-t^{-n})\mu(B)\\ &= \theta\mu(B)\end{aligned}$$

for every $B \in \mathcal{F}_\beta$ and, in consequence,

$$\omega(B - t^{-1}B) \le \frac{1}{2}\omega(B),$$

which obviously imply $\omega(B) \le 2\omega(t^{-1}B)$. So, (1.8.b) can be obtained.

Let us see that our claim is valid. To begin with, we take $y_0 \in \mathbb{R}^n$ such that $|y_0| = \delta r$ with $0 < \delta < \frac{1-\beta}{2}$. Note that $|y_0| < (1-\beta)\beta d(x_0,\Omega^c) < (1-\beta)d(x_0,\Omega^c)$. Then, $B(x_0+y_0,r) \subset B(x_0, d(x_0,\Omega^c))$.

If $B = B(x_0,r) \in \mathcal{F}_{\frac{\beta}{5}}$, noting that $2|y_0| < 4r$, we get

$$\begin{aligned}\int_{\mathbb{R}^n - 5B} |R_j(f+g)|\,dx &\le \int_B \omega(x)\int_{|x-z|>2|y_0|}\left|\frac{x_j - z_j}{|x-z|^{n+1}} - \frac{x_j - z_j + y_j^0}{|x-z+y_0|^{n+1}}\right| dz\,dx\\ &\le C\int_B \omega(x)\,dx,\end{aligned}$$

where $y_0 = (y_1^0,\dots,y_n^0)$, since the kernel of R_j satisfies a Hörmander's condition.

On the other hand,

$$\int_{5B} |R_j(f+g)|\,dx \le \int_{5B} |R_j f|\,dx + \int_{5B} |R_j g|\,dx. \tag{4.4}$$

We can estimate the last integral as follows:

$$\begin{aligned}
\int_{5B} |R_j g|\, dx &\le \int_{B(x_0, 5r+|y_0|)} |R_j f|\, dx \\
&\le \int_{6B-5B} |R_j f|\, dx + \int_{5B} |R_j f|\, dx \\
&\le \int_B \omega(y) \left(\int_{4r \le |x-y| \le 7r} \frac{dx}{|x-y|^n} \right) dy + \int_{5B} |R_j f|\, dx \\
&\le C \int_B \omega\, dx + \int_{5B} |R_j f|\, dx.
\end{aligned}$$

Then, from (1.11.a) and (4.4), we obtain (4.3), since $\tilde{B} = 5B$.

Now, if $B \notin \mathcal{F}_{\frac{\beta}{5}}$, we have $\tilde{B} = \mathcal{N}_\beta(B)$. Note that $2|y_0| < \beta(1-\beta)d(x_0, \Omega^c) \le \beta d(z, \Omega^c)$ for every $z \in B$. So, $B(z,t) \in \mathcal{F}_\beta$ for every $t \in (2|y_0|, \beta d(z, \Omega^c))$, which implies that $x \in \mathcal{N}_\beta(B)$ for every x such that $|x-z| \le 2|y_0|$ for some $z \in B$. In consequence, from Hörmander's condition,

$$\begin{aligned}
&\int_{\mathbb{R}^n - \mathcal{N}_\beta(B)} |R_j(f+g)|\, dx \\
&\qquad \le \int_B \omega(z) \int_{\mathbb{R}^n - \mathcal{N}_\beta(B)} \left| \frac{x_j - z_j}{|x-z|^{n+1}} - \frac{x_j - z_j + y_j^0}{|x-z+y_0|^{n+1}} \right| dx\, dz \\
&\qquad \le \int_B \omega(z) \int_{|x-z| > 2|y_0|} \left| \frac{x_j - z_j}{|x-z|^{n+1}} - \frac{x_j - z_j + y_j^0}{|x-z+y_0|^{n+1}} \right| dx\, dz \\
&\qquad \le C \int_B \omega\, dz.
\end{aligned} \tag{4.5}$$

On the other hand,

$$\begin{aligned}
\int_{\mathcal{N}_\beta(B)} |R_j(f+g)|\, dx &\le \int_{\mathcal{N}_\beta(B)} |R_j f(x)|\, dx + \int_{\mathcal{N}_\beta(B)} |R_j f(x+y_0)|\, dx \\
&\le C \int_B \omega\, dx + \int_{\mathcal{N}_\beta(B)+y_0} |R_j f|\, dx \\
&\le C \int_B \omega\, dx + \int_{(\mathcal{N}_\beta(B)+y_0) - \mathcal{N}_\beta(B)} |R_j f|\, dx.
\end{aligned} \tag{4.6}$$

In order to estimate the last integral, we recall that $x \in \mathcal{N}_\beta(B)$ for every x such that $|x-z| < 2|y_0|$ for some $z \in B$. In addition, appealing to the proof of Lemma 2.3 in [6] (see the proof of Claim 3 on p. 616) once again, we get $d(x_0, \Omega^c) \le cr$ with c not depending on B. This implies $|x + y_0 - z| \le |x-z| + |y_0| \le cr$ for every $x \in \mathcal{N}_\beta(B)$ and $z \in B$. Then, we get

$$\begin{aligned}\int_{(\mathcal{N}_\beta(B)+y_0)-\mathcal{N}_\beta(B)} |R_j f|\,dx &\leq \int_B \omega(z)\left(\int_{2\delta r<|x-z|<Cr} \frac{dx}{|x-z|^n}\right) dz\\ &\leq C\int_B \omega\,dz.\end{aligned}$$

This estimate, together with (4.5) and (4.6) proves (4.3) in this case, concluding the proof of the claim.

Taking into account that $R_j^{\beta,\eta}$ is bounded on $L^p(\Omega, dx)$, $1 < p < \infty$, the reasoning applied in section 3 to prove that (1.8.a) implies (1.8.b) can be used again to prove, this time, (1.11.b). This finishes the proof of the theorem.

5. Proof of Theorem 1.12

We are in the same geometrical setting as in section 3, that is, $X = \mathbb{R}^n$, with the euclidean metric and the Lebesgue measure. The proof of Theorem 1.9 will require some previous technical results.

Lemma 5.1. *Let $0 < \beta < 1$ and $\gamma = \frac{6\beta}{7+\beta}$. Then $S_\gamma(B) \subset E_\beta(\frac{1}{2}B)$ for every $B \in \mathcal{F}_{\frac{\gamma}{10}}$.*

Proof. Let $B = B(x_0, r) \in \mathcal{F}_{\frac{\gamma}{10}}$. Then, for $x \in B$ and $y \in \frac{1}{2}B$, we have

$$|x-y| \leq |x-x_0| + |x_0-y| < r + \frac{1}{2}r < \frac{3}{10}\frac{\gamma}{2}d(x_0,\Omega^c). \tag{5.2}$$

On the other hand

$$\begin{aligned}d(x_0,\Omega^c) &\leq |x_0-z| + d(z,\Omega^c)\\ &< \frac{\gamma}{10}d(x_0,\Omega^c) + d(z,\Omega^c)\end{aligned}$$

for every $z \in B$, which implies

$$(1-\frac{\gamma}{10})d(x_0,\Omega^c) < d(z,\Omega^c).$$

Consequently,

$$\frac{\gamma}{2}d(x_0,\Omega^c) < \frac{\gamma}{2(1-\frac{\gamma}{10})}d(z,\Omega^c) < \frac{5}{9}\gamma d(z,\Omega^c)$$

for every $z \in B$. Then, from this and (5.2), we get

$$|x-y| < \frac{\gamma}{6}d(x,\Omega^c) \tag{5.3}$$

for every $x \in B$ and $y \in \frac{1}{2}B$, and so

$$\begin{aligned}|z-y| &\leq |z-x| + |x-y|\\ &< \gamma d(x,\Omega^c) + \frac{\gamma}{6}d(x,\Omega^c) = \frac{7}{6}\gamma d(x,\Omega^c)\end{aligned} \tag{5.4}$$

for $z \in B(x, \gamma d(x,\Omega^c))$.

From (5.3), we can also obtain $(1-\frac{\gamma}{6})d(x,\Omega^c) < d(y,\Omega^c)$ for every $x \in B$ and $y \in \frac{1}{2}B$. Then, from (5.4) it follows

$$\begin{aligned}|z-y| &< \frac{7}{6}\frac{\gamma}{1-\frac{\gamma}{6}} d(y,\Omega^c)\\ &= \beta d(y,\Omega^c)\end{aligned}$$

for every $x \in B, y \in \frac{1}{2}B$ and $z \in B(x, \gamma d(x,\Omega^c))$, which obviously proves

$$B(x,\gamma d(x,\Omega^c)) \subset B(y,\beta d(y,\Omega^c))$$

for every $x \in B$ and $y \in \frac{1}{2}B$, that is $S_\gamma(B) \subset E_\beta(\frac{1}{2}B)$. □

The next lemma shows an important property of the classes B_p^β. It is not difficult to see that it holds in the more general geometric setting of sections 2 and 3 as well.

Lemma 5.5. *Given $0<\beta<1$ and $p>0$, if $\omega \in B_p^\beta$, then ω satisfies a doubling condition on $\mathcal{F}_\beta$, i.e.: there exists $C>0$ such that $\omega(B) \leq C\omega(\frac{1}{2}B)$ for every $B \in \mathcal{F}_\beta$.*

Proof. Let $B = B(x_0, r) \in \mathcal{F}_\beta$. Then, if $\omega \in B_p^\beta$, we can write

$$\begin{aligned}\omega(B) &\leq \int_{\frac{r}{2}<|x_0-y|<r} \left(\frac{r}{|x_0-y|}\right)^{n+p} \omega(y)\,dy + \omega\left(\frac{1}{2}B\right)\\ &\leq C\Big(\frac{r}{2}\Big)^{n+p} \int_{S_\beta(\frac{1}{2}B)-\frac{1}{2}B} \frac{\omega(y)}{|x_0-y|^{n+p}}\,dy + \omega\left(\frac{1}{2}B\right)\\ &\leq C\omega\left(\frac{1}{2}B\right),\end{aligned}$$

and the lemma is done. □

Now, we introduce a definition that will be useful to proof other properties of B_p^β.

Definition 5.6. *Given $0<\beta<1$ and $p>0$, we say that a weight ω belongs to $\tilde{B}_p^\beta$ whenever*

$$\omega(B) \leq Ct^{n+p-\varepsilon}\omega\left(\frac{1}{t}B\right)$$

for every $B \in \mathcal{F}_\beta, t>1$ and some constants $C>0$ and $\varepsilon>0$ independent of B and t.

The following couple of technical results will allow us to connect the classes $\tilde{B}_p^\beta$ and B_p^β.

Lemma 5.7. *Let $M>0$ and φ be a non decreasing and non negative function defined on $(0,M]$ such that*

$$\int_t^M \frac{\varphi(s)}{s^{r+1}}\,ds \leq C_1\frac{\varphi(t)}{t^r} \quad \textit{and} \quad \varphi(t) \leq C_2\varphi\left(\frac{1}{2}t\right) \tag{5.8}$$

for every $t \in (0, M]$ and some positive constants C_1, C_2 and $r > 0$, not depending on t. Then, the function $g(t) = \frac{\varphi(t)}{t^r}$ is quasi-decreasing on $(0, M]$ (i.e.: there exists $C > 0$ such that $g(t_2) \leq Cg(t_1)$ for $t_1 \leq t_2$).

Proof. It follows easily from the conditions on φ. $\square$

Lemma 5.9. *Let φ be a function as in Lemma 5.7. Then, the condition* (5.8) *is equivalent to each one of the following statements.*

(5.9.a) There exists $a > 1$ such that $\varphi(t) \leq \frac{a^r}{2}\varphi(\frac{t}{a})$ for every $t \in (0, M]$

(5.9.b) There exist positive constants C and ε such that $\varphi(t) \leq C\theta^{r-\varepsilon}\varphi(\frac{t}{\theta})$ for every $\theta \geq 1$ and $t \in (0, M]$.

Proof. The lemma can be proved following the same ideas with obvious changes of those applied in the proof of Lemma (3.3) in [5]. $\square$

Lemma 5.10. *Let $\alpha, \beta \in (0, 1)$ and $p > 0$. Then $\tilde{B}_p^\alpha = \tilde{B}_p^\beta$.*

Proof. Note that each weight ω in $\tilde{B}_p^\alpha$ is doubling on $\mathcal{F}_\alpha$ and, in consequence, on $\mathcal{F}_\gamma$ for every $\gamma \in (0, 1)$. The lemma is an immediate consequence of this fact. $\square$

Lemma 5.11. *Let $p > 0$ and $\beta \in (0, 1)$. Then $B_p^\beta \subset \tilde{B}_p^\beta$.*

Proof. Let $\omega \in B_p^\beta$. Taking into account Lemma 5.5, it is not difficult to see that there exists $C > 0$ such that $\omega(B) \leq C\omega(B - \frac{1}{2}B)$ for every $B \in \mathcal{F}_{\frac{2}{5}\beta}$. With this in mind, we denote $\beta_0 = \frac{2}{5}\beta$ and take $B = B(x_0, r) \in \mathcal{F}_{\frac{\beta_0}{2}}$. Then, for $m \in \mathbb{N}$ satisfying $\frac{\beta_0}{2^{m+1}}d(x_0, \Omega^c) \leq r < \frac{\beta_0}{2^m}d(x_0, \Omega^c)$

$$
\begin{aligned}
C\frac{\omega(B)}{r^{n+p}} &\geq \int_{S_\beta(B)-B} \frac{\omega(y)}{|x_0 - y|^{n+p}}\, dy \\
&\geq \sum_{K=0}^{m-1} \frac{\omega\left(B\left(x_0, \frac{\beta_0 d(x_0,\Omega^c)}{2^K}\right) - B\left(x_0, \frac{\beta_0 d(x_0,\Omega^c)}{2^{K+1}}\right)\right)}{\left(\frac{\beta_0 d(x_0,\Omega^c)}{2^K}\right)^{n+p}} \\
&\geq \tilde{C}\sum_{K=0}^{m-1} \int_{\frac{\beta_0 d(x_0,\Omega^c)}{2^{K+1}}}^{\frac{\beta_0 d(x_0,\Omega^c)}{2^K}} \frac{\omega(B(x_0, u))}{u^{n+p+1}}\, du \\
&\geq \tilde{C}\int_r^{\frac{\beta_0}{2}d(x_0,\Omega^c)} \frac{\omega(B(x_0, u))}{u^{n+p+1}}\, du,
\end{aligned}
$$

with C and $\tilde{C}$ independent of r and x_0. This inequality and Lemma 5.9 imply $\omega \in \tilde{B}_p^{\frac{\beta_0}{2}}$. Finally, Lemma 5.10 concludes the proof. $\square$

The following lemma shows that the classes B_p^β, like $\tilde{B}_p^\beta$ and A_p^β, are independent of β.

Lemma 5.12. *Let $\alpha, \beta \in (0,1)$ and $p > 0$. Then $B_p^\alpha = B_p^\beta$.*

Proof. Let $\omega \in B_p^\alpha$. If $\beta < \alpha$, it is obvious that $\omega \in B_p^\beta$, since $\mathcal{F}_\beta \subset \mathcal{F}_\alpha$ and $S_\beta(B) \subset S_\alpha(B)$ for every $B \in \mathcal{F}_\beta$. Let us assume $\alpha < \beta$. From Lemma 2.3, with $\varepsilon_0 = 1$ and $\lambda = \beta$, we get $\varepsilon_1 \frac{\alpha}{\beta}$ and $\varepsilon_2 \in (0,1)$ such that $0 < \beta + \varepsilon_2 < 1$ and $S_\beta(B) \subset B(x_0, (\beta+\varepsilon_2) d(x_0, \Omega^c))$ for every $B = B(x_0, r) \in \mathcal{F}_{\varepsilon_1 \beta}$. Then, for such balls, we have

$$
\begin{aligned}
r^{n+p} &\int_{S_\beta(B)-B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \\
&\leq r^{n+p} \int_{B(x_0,(\beta+\varepsilon_2)d(x_0,\Omega^c))-B(x_0,\varepsilon_1\beta d(x_0,\Omega^c))} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \\
&\quad + r^{n+p} \int_{B(x_0,\varepsilon_1\beta d(x_0,\Omega^c))-B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \\
&\leq \frac{r^{n+p}}{(\varepsilon_1 \beta d(x_0,\Omega^c))^{n+p}} \omega(B(x_0,(\beta+\varepsilon_2)d(x_0,\Omega^c))) \\
&\quad + r^{n+p} \int_{S_\alpha(B)-B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy, \\
&\leq C \left(\frac{r^{n+p}}{(\varepsilon_1\beta d(x_0,\Omega^c))^{n+p}} \left(\frac{(\beta+\varepsilon_2) d(x_0,\Omega^c)}{r} \right)^{n+p-\varepsilon} + 1 \right) \omega(B), \\
&\leq C\omega(B),
\end{aligned}
\tag{5.13}
$$

where we have applied Lemmas 5.10, 5.11 and the hypothesis on ω.

If $B \in \mathcal{F}_\beta - \mathcal{F}_{\varepsilon_1\beta}$, since $S_\beta(B) \subset \mathcal{N}_\beta(B)$, it is an easy consequence of Lemma 3.1 in [6] that $\omega(S_\beta(B)) \leq C\omega(B)$ with C independent of B. This implies

$$
r^{n+p} \int_{S_\beta(B)-B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \leq C\omega(B),
$$

which, together with (5.13), proves $\omega \in B_p^\beta$. $\square$

Our last result is the converse to Lemma 5.11.

Lemma 5.14. *Let $p > 0$ and $\beta \in (0,1)$. Then $\tilde{B}_p^\beta \subset B_p^\beta$.*

Proof. Given $\beta \in (0,1)$, we know from Lemma 2.3 that we can choose constants θ_1 and θ_2 such that $S_\beta(B) \subset B(x_0, \theta_1 d(x_0,\Omega^c))$ for every ball $B = B(x_0,r) \in \mathcal{F}_{\theta_2}$ and $\theta_2 < \beta < \theta_1 < 1$. Then, we can obtain

$$
\begin{aligned}
r^{n+p} \int_{S_{\theta_2}(B)-B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy &\leq r^{n+p} \int_{S_\beta(B)-B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \\
&\leq \omega(B(x_0,\theta_1 d(x_0,\Omega^c))) \\
&\leq C\omega(B(x_0,r))
\end{aligned}
\tag{5.15}
$$

for every ball $B = B(x_0,r) \in \mathcal{F}_{\theta_2} - \mathcal{F}_{\frac{\theta_2}{5}}$. Note that, in addition, we can chose θ_1 such that $5\theta_1 \in (5\beta, 1)$ whenever $\beta < \frac{1}{5}$. Then, taking $B = B(x_0,r) \in \mathcal{F}_{\frac{\theta_2}{5}}$, for

$K_0 \in \mathbb{N}$ such that $5^{K_0} r \leq \theta_2 d(x_0, \Omega^c) < 5^{K_0+1} r$, we get $S_{\theta_2}(B) \subset S_{\theta_2}(5^{K_0}B) \subset B(x_0, \theta_1 d(x_0, \Omega^c)) \subset B(x_0, \frac{5\theta_1}{\theta_2} 5^{K_0} r)$ and $B(x_0, \frac{5\theta_1}{\theta_2} 5^{K_0} r) \in \mathcal{F}_{5\theta_1}$. With this in mind, we can proceed as follows:

$$\begin{aligned} r^{n+p} \int_{S_{\theta_2}(B)-B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy &\leq r^{n+p} \int_{S_{\theta_2}(5^{K_0}B)-5^{K_0}B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \\ &\quad + r^{n+p} \int_{5^{K_0}B-B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \\ &= I + II. \end{aligned} \tag{5.16}$$

Let us estimate I. Assuming $\frac{5\theta_1}{\theta_2} B \subset 5^{K_0} B$, we get

$$\begin{aligned} I &\leq r^{n+p} \int_{\frac{5\theta_1}{\theta_2}5^{K_0}B - \frac{5\theta_1}{\theta_2}B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \\ &\leq r^{n+p} \sum_{K=0}^{K_0-1} \int_{\frac{5\theta_1}{\theta_2}5^{K+1}B - \frac{5\theta_1}{\theta_2}5^K B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \\ &\leq C r^{n+p} \sum_{K=0}^{K_0-1} \left(\frac{5\theta_1}{\theta_2} 5^K r\right)^{-n-p} \omega\left(\frac{5\theta_1}{\theta_2} 5^{K+1} B\right) \\ &\leq C \sum_{K=0}^{K_0-1} \left(\frac{5\theta_1}{\theta_2} 5^K\right)^{-n-p} \left(\frac{5\theta_1}{\theta_2} 5^{K+1}\right)^{n+p-\varepsilon} \omega(B) \\ &\leq C \left(\frac{\theta_2}{5\theta_1}\right)^{\varepsilon} \omega(B) \sum_{K=0}^{\infty} 5^{-K\varepsilon} \\ &\leq C\omega(B), \end{aligned}$$

where we applied that $\omega \in \tilde{B}_p^\beta$.

On the other hand, if $5^{K_0} B \subset \frac{5\theta_1}{\theta_2} B$, we obtain

$$I \leq r^{n+p} \int_{\frac{5\theta_1}{\theta_2}5^{K_0}B - \frac{5\theta_1}{\theta_2}B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy + r^{n+p} \int_{\frac{5\theta_1}{\theta_2}B - B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy.$$

The first integral on the right side can be estimated as before, while the second one is clearly lesser than a constant times $\omega(B)$. Let us see II.

$$\begin{aligned} II &\leq r^{n+p} \sum_{K=0}^{K_0-1} \int_{5^{K+1}B-5^K B} \frac{\omega(y)}{|x_0-y|^{n+p}}\, dy \\ &\leq C \sum_{K=0}^{K_0-1} 5^{-K(n+p)} 5^{K(n+p-\varepsilon)} \omega(B) \\ &\leq C\omega(B), \end{aligned}$$

where we used once again that $\omega \in \tilde{B}_p^\beta$.

Finally, from (5.15) and (5.16), the estimates of I and II, and Lemma 5.12 we get $\omega \in B_p^\beta$. $\square$

Now, we are in position to prove the main result of this section.

Proof of Theorem 1.9. Let $B = B(x_0, r) \in \mathcal{F}_{\frac{\beta}{8}}$. Note that $2B \subset E_{\frac{\beta}{2}}(B) = \bigcap_{x\in B} B(x, \frac{\beta}{2} d(x, \Omega^c))$. In addition, if $y \in B(x, \frac{\beta}{2} d(x, \Omega^c))$ for some $x \in \Omega$, we get $\eta\left(\frac{|x-y|}{\beta d(x,\Omega^c)}\right) = 1$. Then, following an analogous reasoning to that used in the proof of Theorem 2 of [4] (see p. 533) we can obtain

$$r^{n+1} \int_{E_{\frac{\beta}{2}}(B)-B} \frac{\omega(y)}{|x-y|^{n+1}}\, dy \leq C\omega(B),$$

with C independent of B. Then, since $\mathcal{F}_{\frac{\gamma}{10}} \subset \mathcal{F}_{\frac{\beta}{8}}$ for $\gamma = \frac{6\beta}{14+\beta}$, from Lemma 5.1 it follows

$$r^{n+1} \int_{S_\gamma(B)-B} \frac{\omega(y)}{|x-y|^{n+1}}\, dy \leq C\omega(B)$$

for every $B \in \mathcal{F}_{\frac{\gamma}{10}}$. Consequently, $\omega \in B_1^{\frac{\gamma}{10}}$ and, from Lemma 5.12, $\omega \in B_1^\beta$.

On the other hand, if $B \in \mathcal{F}_{\frac{\beta}{5}}$, by reasoning as in [4] (see the proof of Theorem 2 there) it can be proved that

$$\begin{aligned} \int_{5B} |R_j^{\beta,\eta}(\omega \mathcal{X}_B)|\, dx &\leq C \int_{5B} \omega\, dx \\ &\leq C \int_B \omega\, dx \end{aligned} \tag{5.17}$$

where the last inequality follows from the fact that ω is doubling.

In the case $B \in \mathcal{F}_\beta - \mathcal{F}_{\frac{\beta}{5}}$, we know, from Lemma 2.3 in [6], that $\mathcal{N}_\beta(B)$ can be covered with a finite number of balls, say $P_1, \ldots, P_m$, belonging to $\mathcal{F}_{\frac{\beta}{5}}$. It is clear that for each P_i we can pick a ball $B_i \in \mathcal{F}_\beta - \mathcal{F}_{\frac{\beta}{5}}$ such that $P_i \bigcap B_i \neq \emptyset$ and $B_i \bigcap B \neq \emptyset$. If we choose balls $P_i^* \in \mathcal{F}_\beta - \mathcal{F}_{\frac{\beta}{2}}$ concentric with P_i, we get $P_i^* \subset \mathcal{N}_\beta(B_i)$ and $B_i \subset \mathcal{N}_\beta(B)$ for each i. Then, we have

$$\begin{aligned} \int_{\mathcal{N}_\beta(B)} |R_j^{\beta,\eta}(\omega\mathcal{X}_B)|\, dx &\leq \sum_{i=1}^m \int_{P_i^*} |R_j^{\beta,\eta}(\omega\mathcal{X}_B)|\, dx \\ &\leq \sum_{i=1}^m \omega(P_i^*) \left[R_j^{\beta,\eta}(\omega\mathcal{X}_B)\right]_{\mathrm{BMO}_\omega^\beta} \\ &\leq C \sum_{i=1}^m \omega\left(\mathcal{N}_\beta(B_i)\right) \\ &\leq C \sum_{i=1}^m \omega(B_i) \end{aligned}$$

$$\begin{aligned} &\leq C \sum_{i=1}^{m} \omega\left(\mathcal{N}_\beta(B)\right) \\ &\leq C\omega(B), \end{aligned}$$

which, together with (5.17) and Theorem 1.8, proves $\omega \in A_\infty^\beta$ and finishes the proof. □

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Received: November 3, 2022
Accepted: March 22, 2023