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A TRIBUTE TO POLA HARBOURE: ISOPERIMETRIC INEQUALITIES AND THE HMS EXTRAPOLATION THEOREM

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Abstract. We give a simpler proof of the Gagliardo estimate with a measure obtained by Franchi, Pérez, and Wheeden [*Proc. London Math. Soc. (3)* **80** no. 3 (2000), 665–689], and improved by Pérez and Rela [*Trans. Amer. Math. Soc.* **372** no. 9 (2019), 6087–6133]. This result will be further improved using fractional Poincaré type inequalities with the extra bonus of Bourgain–Brezis–Mironescu as done by Hurri-Syrjänen, Martínez-Perales, Pérez, and Vähäkangas [*Internat. Math. Res. Notices* (2022), rnac246] with a new argument. This will be used with the HMS extrapolation theorem to get L^p type result.

1. The isoperimetric inequality and extrapolation theory

It is a great pleasure for us to dedicate this article to Eleonor Harboure, Pola, who played a central role in the development of modern Harmonic Analysis in Argentina. The first author is deeply grateful for her kind support during early stages of his career. Both authors want to stress how influential the work of Pola was to the mathematical community. This paper is also a tribute to the extrapolation theorem of Pola, R. Macías, and C. Segovia which was published in the American Journal of Mathematics [21] (see also [20]). See Theorem 2.1 in Section 2 for an updated version. We will refer to it as the *HMS extrapolation theorem*. Thanks to this result we can complete some of the main results obtained in [32] in the classical setting. A fractional counterpart with the Bourgain–Brezis–Mironescu gain will be obtained in the line of results as derived in [22] or [3].

The HMS extrapolation theorem was inspired by the classical extrapolation theorem of Rubio de Francia [8, 10, 18].

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1.1. **The classical context.** A *first* purpose of this paper is to give a simple proof of the following extension of the celebrated Gagliardo's inequality which can be seen as an *extension of the classical isoperimetric inequality* although the best constant was not obtained by Gagliardo.

$\mathcal{Q}$ will denote throughout the paper the family of cubes Q, i.e. a cartesian product of n intervals of the same length $\ell(Q)$ in $\mathbb{R}^n$ and

$$f_Q := \fint_Q f(x)\,dx$$

is the average of f over the cube Q. M will always denote the maximal function operator:

$$Mf(x) = \sup_{\mathcal{Q}\ni Q\ni x} \fint_Q |f(y)|\; dy,$$

The centered version with respect to euclidean balls of this operator is defined as

$$M^c f(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\; dy.$$

Since we are considering the euclidean space $\mathbb{R}^n$ endowed with the Lebesgue measure, both maximal operators defined above are pointwise comparable, up to a dimensional constant.

The main principle of this paper is that Theorem 1.1 below combined with the HMS extrapolation theorem yields the classical $L^p \to L^{p^*}$ Sobolev embedding and the modern one with the right class of weights.

Theorem 1.1. *Let μ be any measure in $\mathbb{R}^n$, $n \geqslant 2$, then there exists a dimensional constant c_n such that for any cube $Q \in \mathcal{Q}$ and any Lipschitz function f, we have*

$$\|f - f_Q\|_{L^{n',\infty}(Q,d\mu)} \leqslant c \int_Q |\nabla f(x)| (M^c(\chi_Q \mu)(x))^{\frac{1}{n'}}\; dx. \tag{1.1}$$

As a consequence,

$$\|f - f_Q\|_{L^{n'}(Q,d\mu)} \leqslant c \int_Q |\nabla f(x)| (M^c(\chi_Q \mu)(x))^{\frac{1}{n'}}\; dx, \tag{1.2}$$

and hence,

$$\left(\int_{\mathbb{R}^n} |f(x)|^{n'} d\mu(x)\right)^{\frac{1}{n'}} \leqslant c_n \int_{\mathbb{R}^n} |\nabla f(x)| (M^c \mu(x))^{\frac{1}{n'}}\; dx, \tag{1.3}$$

where f is a Lipschitz function with compact support.

Some vector-valued extensions of these results have been obtained in [31] even with the sharp isoperimetric constant (1.5) when the global situation is considered.

Theorem 1.1 is implicit in [16] in a different more general context and was made explicit and extended in [32]. Inequality (1.2) follows from (1.1) by using the well known *truncation method*, also called the "weak implies strong" argument, which will be used several times in this article. This method can be found in Maz'ya [27]. See also [32] and the very nice survey article of the method [19]. Theorem 1.1 has been extended in [29] in different directions using a different direct approach, in particular the truncation method is avoided.

Certainly, (1.1) is motivated by the well known Fefferman-Stein inequality [13],

$$\|Mf\|_{L^{1,\infty}(d\mu)} \leqslant c_n \|f\|_{L^1(M\mu)}$$

from which we deduce, for $p \in (1, \infty)$, that

$$\|Mf\|_{L^p(d\mu)} \leqslant c_n\, p' \, \|f\|_{L^p(M\mu)},$$

where μ is any general measure. This is probably the main motivation to define the A_1 class of weights, namely those weights w such that

$$Mw \leqslant cw,$$

being the best constant $[w]_{A_1}$ which is a number bigger or equal than one. A relevant well known class of A_1 weights is given by the celebrated Coifman-Rochberg theorem [5], that is for $\delta \in (0,1)$, then $(M\mu)^\delta \in A_1$ and

$$[(M\mu)^\delta]_{A_1} \leqslant \frac{c_n}{1-\delta}, \tag{1.4}$$

whenever the function $M\mu$ is finite almost everywhere. Observe that the weight defined by $(M^c(\chi_Q \mu)(x))^{\frac{1}{n'}}$ in Theorem 1.1 satisfies the A_1 condition by (1.4) since it is comparable to $(M(\chi_Q \mu)(x))^{\frac{1}{n'}}$ up to a dimensional constant.

Inequality (1.3) with the choice of $d\mu = dx$ is part of the large family of Sobolev type inequalities. However in this case, namely when $p = 1$, it is not due to Sobolev but to E. Gagliardo [17]. Gagliardo's proof did not, however, give the best possible constant. This was obtained by Maz'ya [27] and, independently, by H. Federer and W. H. Fleming [12] yielding an extension of the isoperimetric inequality in $\mathbb{R}^n$,

$$\alpha_n |\Omega|^{\frac{n-1}{n}} \leqslant \mathcal{H}^{n-1}(\partial\Omega), \tag{1.5}$$

which holds for any sufficiently smooth domain Ω where $\alpha_n = n\,|B_1(0)|^{\frac{1}{n}}$. As usual, $|\Omega|$ denotes the n-dimensional volume of Ω and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure. Namely, (1.5) is equivalent to

$$\alpha_n \left(\int_{\mathbb{R}^n} |f(x)|^{n'} dx \right)^{\frac{1}{n'}} \leqslant \int_{\mathbb{R}^n} |\nabla f(x)|\, dx, \tag{1.6}$$

for any Lispchitz function with compact support, being α_n the best possible constant. This shows that (1.3) can be seen as a non-sharp version of (1.6) from which we have

Corollary 1.2. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ and let μ be a measure. Then,*

$$\mu(\Omega)^{\frac{n-1}{n}} \leqslant c_n \int_{\partial\Omega} (M^c\mu)^{\frac{1}{n'}} \, d\mathcal{H}^{n-1},$$

This result can be seen as an extension of the main result obtained by David-Semmes in [9] and improved in [36].

We finish this section with the following natural conjecture.

Conjecture 1.3. *Let Ω and μ be as before. Then, the following estimate holds for any Lispchitz function with compact support,*

$$\alpha_n \left(\int_{\mathbb{R}^n} |f(x)|^{n'} d\mu(x) \right)^{\frac{1}{n'}} \leqslant \int_{\mathbb{R}^n} |\nabla f(x)| (M^c \mu(x))^{\frac{1}{n'}} \, dx,$$

where α_n the best possible constant.

We remit to Osserman [30], Ziemer [37] or Talenti [35] for more information about this topic.

1.2. **The influence of the extrapolation theory.** The appearance of the factor $(M\mu(x))^{\frac{1}{n'}}$ in Theorem 1.1 gives rise in a natural way to the definition of the $A_{1,n'}$ class of weights which will play a central role in this paper. More precisely, it will be the starting point in the main applications of the HMS extrapolation theorem. This class is defined as the weights w, such that

$$M(w^{n'}) \leqslant c w^{n'}$$

Observe that $A_{1,n'}$ is a subclass of A_1. Also observe that $A_{1,n'}$ is part of a larger family of weights denoted by A_{p,p^*} and defined by,

$$[w]_{A_{p,p^*}} = \sup_Q \left(\fint_Q w^{p^*} \right) \left(\fint_Q w^{-p'} \right)^{\frac{p^*}{p'}} < \infty. \tag{1.7}$$

p^* is the usual Sobolev exponent,

$$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}.$$

This condition was introduced by B. Muckenhoupt and R. Wheeden in [28]. We remit to Section 2 for more information about this class of weights.

The *second* purpose of this paper is to use Theorem 1.1 combined with the HMS extrapolation theorem to derive following new local result.

Corollary 1.4. *Let $p \in [1, n)$ and let $w \in A_{p,p^*}$. Then there exists a constant $c_{n,p}$ such that for any cube Q,*

$$\|w(f - f_Q)\|_{L^{p^*}(Q,dx)} \leqslant c_{n,p} [w]_{A_{p,p^*}}^{\frac{1}{n'}} \left(\int_Q |w \nabla f|^p dx \right)^{\frac{1}{p}}.$$

As a consequence we have the global estimate

$$\|wf\|_{L^{p^*}(\mathbb{R}^n)} \leqslant c_{n,p} \, [w]_{A_{p,p^*}}^{\frac{1}{n'}} \, \|w \nabla f\|_{L^p(\mathbb{R}^n)}.$$

The main novelty of this result lies in the fact that the exponent of the constant in front $[w]_{A_{p,p^*}}^{\frac{1}{n'}}$ is the sharpest possible.

Another nice improvement of this result using mixed $A_1 - A_\infty$ bounds can be found in [33].

Of course, this result contains as well the very well known Poincaré–Sobolev inequalities on cubes (see (2.1) below). Being a bit more precise, it is necessary to use the sharp version of the HMS extrapolation theorem obtained in [25] to derive this sharp result. In other words, a good "weighted" initial estimate like (1.1) carries all the relevant information thanks to the HMS extrapolation theorem.

A (p^*, p) extension of (1.1) and hence of (1.2) is contained in the following theorem which can be found in [32].

Theorem 1.5. *Let w be a weight in $\mathbb{R}^n$, $n \geqslant 2$. Then if $1 \leqslant p < n$ we have that*

$$\left(\int_Q |f - f_Q|^{p^*} w dx\right)^{\frac{1}{p*}} \leqslant C \left(\int_Q \left(\frac{|\nabla f|}{w} M^c(w\chi_Q)^{\frac{1}{n'}}\right)^p w dx\right)^{\frac{1}{p}}.$$

This estimate is interesting on its own since nothing is assumed on the weight w. However, it does not produce the class of weights $A_{p,p*}$ from Corollary 1.4 since we are clearly restricted to the case $A_{1,n'}$ due to the presence of $(M^c w)^{\frac{1}{n'}}$. There should be a proof more in the spirit of the proof of (1.2) (see Section 3).

An interesting observation is that a similar result holds for higher order derivatives but *only* in the weak case, namely

$$\|f - P_Q f\|_{L^{\frac{n}{n-m},\infty}(Q,d\mu)} \leqslant c \int_Q |\nabla^m f(x)| (M^c(\chi_Q \mu)(x))^{\frac{n-m}{n}} \, dx, \tag{1.8}$$

where $P_Q f$ is an appropriate polynomial of order $m-1$. This estimate seems not be known. Recall that the polynomial $P_Q f$ is an optimal special polynomial in the sense that

$$\inf_{\pi \in \mathcal{P}_m} \left(\frac{1}{|Q|}\int_Q |f - \pi|^p\right)^{1/p} \approx \left(\frac{1}{|Q|}\int_Q |f - P_Q f|^p\right)^{1/p},$$

where $\mathcal{P}_m$ is the family of polynomials of degree lees or equal than $m-1$. Roughly speaking $P_Q f$ is the Taylor polynomial of f on Q.

The proof of (1.8) is the same as in the case $m = 1$ using an extension of the representation formula (2.3),

$$|f(x) - P_Q f(x)| \leqslant c_{n,m} \, I_m(|\nabla^m f| \chi_Q)(x) \qquad x \in Q.$$

See [1] and [26, p. 45]. The drawback of this approach is that the truncation method cannot be applied in the case of higher order derivatives. However, we believe that the strong result is true and we conjecture it.

Conjecture 1.6. *There is a constant c depending on n, m such that*

$$\inf_{\pi \in P_m} \|f - \pi\|_{L^{\frac{n}{n-m}}(Q,d\mu)} \leqslant C \int_Q |\nabla^m f(x)| \, (M^c(\chi_Q)^{\frac{n-m}{n}} \, dx.$$

Recall that for the usual case $d\mu = dx$, the result holds by iteration from the case $m = 1$ which seems not feasible for this general situation.

1.3. **The new millennium's, the influence of BBM.** Theorem 1.1 was improved in [22] to the context of the influential work by Bourgain, Brezis and Mironescu in [2, Theorem 2.10]. The *third* purpose of this paper is to give a different proof of this [22] result which is the following.

Theorem 1.7 ([22]). *Let μ be a measure in $\mathbb{R}^n$ and let $0 < \delta < 1$. Then there exists a dimensional constant c_n such that for any $Q \in \mathcal{Q}$ and any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*

$$\begin{aligned}\left(\int_Q |f(x) - f_Q|^{\frac{n}{n-\delta}}\, d\mu(x)\right)^{\frac{n-\delta}{n}} \qquad (1.9)\\ \leqslant c_n\,(1-\delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}}\, dy\, M^c(\chi_Q \mu)(x)^{\frac{n-\delta}{n}}\, dx.\end{aligned}$$

Our proof is based on a similar idea of the proof of Theorem 1.1.

This result improves the one in [22] since it is valid for general measures μ instead of (general) weights and the smaller centered maximal function of the right hand side instead of the non-centered one.

As a corollary we obtain the following global fractional isoperimetric inequality follows with the $(1-\delta)$ gain.

Corollary 1.8. *Let μ be a measure in $\mathbb{R}^n$ and let $0 < \delta < 1$. There is a positive dimensional constant c_n such that,*

$$\left(\int_{\mathbb{R}^n} |f|^{\frac{n}{n-\delta}}\, w\right)^{\frac{n-\delta}{n}} \leqslant c_n\,(1-\delta) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}}\, dy\, (Mw(x))^{\frac{n-\delta}{n}}\, dx,$$

for an appropriate class of functions f.

As before some vector-valued extensions of these results have been obtained in [31].

We remark that Theorem 1.7 yields Theorem 1.1. Indeed, it is shown in [29] (also [23]) that for any $\delta \in (0,1)$,

$$\int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} dy\, w(x) dx \leqslant \frac{c_n \ell(Q)^{1-\delta}}{\delta(1-\delta)} \int_Q |\nabla f(x)|\, Mw(x)\, dx.$$

Then, combining this result with Theorem 1.7, using that $M^c(\chi_Q \mu)(x)^{\frac{n-\delta}{n}}$ is an A_1 with constant $\frac{c_n}{\delta}$ (see (1.4)) and letting $\delta \to 1+$ yields estimate (1.2).

In this paper we give an alternative proof of Theorem 1.7 using the same key idea as in the proof of (1.2) shown here. First, the following appropriate and key weak type estimate

$$\|f - f_Q\|_{L^{\frac{n}{n-\delta},\infty}(Q,d\mu)} \leqslant c_n\,(1-\delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}}\, dy\, M\mu(x)^{\frac{n-\delta}{n}}\, dx.$$

is proved in (3.8). This combined with a variant of the truncation method for the gradient, called the "fractional truncation method", see [11, Theorem 4.1] and [4, Proposition 2.14] yields the strong type estimate (1.9). We remit to [22] for further details.

As a consequence of Theorem 1.7 we derive a fractional version of Corollary 1.4. The key point is to use an adaptation of the HMS theorem as done in the proof of Corollary 1.4 due to the special structure of the right fractional L^p version of the right-hand side. Specifically, the following result can be obtained.

Theorem 1.9. *Let $p \in [1, \frac{n}{\delta})$ and let $w \in A_{p,p_\delta^*}$. Then there exists a constant $c_{n,p}$ such that for any cube Q,*

$$\|w(f-f_Q)\|_{L^{p_\delta^*}(Q,dx)} \leqslant c_{n,p}(1-\delta)^{\frac{1}{p}}[w]_{A_{p,p_\delta^*}}^{\frac{n-\delta}{n}} \left(\int_Q\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\delta}}\,dy\,wdx\right)^{\frac{1}{p}}.$$

As a consequence, we have the global estimate

$$\|wf\|_{L^{p^*}(\mathbb{R}^n)} \leqslant c_{n,p}\,(1-\delta)^{\frac{1}{p}}[w]_{A_{p,p_\delta^*}}^{\frac{n-\delta}{n}} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\delta}}\,dy\,wdx\right)^{\frac{1}{p}}.$$

2. Some preliminaries and their history

- The *rough* prehistory:

As already mentioned, Theorem 1.5 yields

$$\left(\frac{1}{|Q|}\int_Q |f-f_Q|^{p^*}\right)^{1/p^*} \leqslant c\,\ell(Q),\left(\frac{1}{|Q|}\int_Q |\nabla f|^p\right)^{1/p} \qquad Q\in\mathcal{Q}. \tag{2.1}$$

from which we derive the global result usually called the Gagliardo–Nirenberg–Sobolev inequalities,

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leqslant c_n\,\|\nabla f\|_{L^p(\mathbb{R}^n)}, \qquad f\in C_c^1(\mathbb{R}^n), \tag{2.2}$$

when $1 \leqslant p < n$. Extensions and new variations of this estimate can be found in [6, 7].

The most common approach to prove (2.1) is based on the following pointwise estimate which controls the oscillation of the function by the fractional integral (see [24, 34])

$$|f(x)-f_Q| \leqslant c_n\, I_1(|\nabla f|\chi_Q)(x). \tag{2.3}$$

It is an interesting fact that (2.3) is equivalent to the following averaged result,

$$\frac{1}{|Q|}\int_Q |f-f_Q| \leqslant c_n\,\frac{\ell(Q)}{|Q|}\int_Q |\nabla f|,$$

as shown first in [15] (see an extension in [32], Theorem 11.3). Then the proof of (2.1) in the range $1<p<n$ is based on the boundedness

$$I_1 : L^p(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n) \tag{2.4}$$

which is a well known classical result (see the recent monograph [24]), being the boundedness false when $p=1$, $1^* = \frac{n}{n-1} = n'$. I_1 is the Riesz potential operator of order $\alpha = 1$ given by the expression

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad 0<\alpha<n.$$

An interesting fact is that the strong inequality (2.1) with $p=1$, namely

$$\left(⨍_Q |f(x)-f_Q|^{n'}\,dx\right)^{\frac{1}{n'}} \leqslant c_n\,\ell(Q)\left(⨍_Q |\nabla f(x)|\,dx\right),$$

follows from the weak endpoint boundedness of I_1, namely

$$I_1 : L^1(\mathbb{R}^n) \to L^{n',\infty}(\mathbb{R}^n).$$

To obtain the corresponding strong inequality (2.1) we may use the (already mentioned) truncation method of Maz'ya [27].

• The *rough* weighted theory:

A relevant extension of (2.4) and hence of (2.1) and (2.2) in the case $p > 1$ was obtained by B. Muckenhoupt and R. Wheeden [28]. They showed in this paper that I_1 satisfies weighted bounds of the form

$$\|wI_1 f\|_{L^{p*}(\mathbb{R}^n)} \leqslant c\,\|wf\|_{L^p(\mathbb{R}^n)} \tag{2.5}$$

if and only if $w \in A_{p,p*}$ already defined in (1.7),

$$[w]_{A_{p,p*}} = \sup_Q \left(\fint_Q w^{p*}\right)\left(\fint_Q w^{-p'}\right)^{\frac{p*}{p'}} < \infty.$$

Again, at the endpoint $p = 1$ only the weak boundedness holds, and Muckenhoupt and Wheeden proved that

$$\|I_1 f\|_{L^{n',\infty}(w^{n'})} \leqslant c\,\|wf\|_{L^1(\mathbb{R}^n)}$$

holds if and only if $w \in A_{1,n'}$, i.e.

$$\left(\fint_Q w^{n'}\right) \leqslant c \inf_Q (w^{n'})$$

where the smallest constant c is denoted as $[w]_{A_{1,n'}}$.

$$\|wf\|_{L^{p*}(\mathbb{R}^n)} \leqslant c\,\|w\nabla f\|_{L^p(\mathbb{R}^n)}, \qquad f \in C_c^\infty(\mathbb{R}^n).$$

• Precise weighted estimates: *The HMS theorem*:

Now, besides its dependence on the dimension and p, the constant c in (2.5) also depends on the constant $[w]_{A_{p,p*}}$.

The optimal weighted bound for I_1 was proven in [25]

$$\|wI_1 f\|_{L^{p*}(\mathbb{R}^n)} \leqslant c_p\,[w]_{A_{p,p*}}^{\frac{1}{n'}\max\{1,\frac{p'}{p*}\}}\,\|wf\|_{L^p(\mathbb{R}^n)}, \qquad 1 < p < n,$$

which leads to the following improvement of (2.1) whenever $p \in (1, n)$,

$$\|w(f - f_Q)\|_{L^{p*}(Q,dx)} \leqslant c_{n,p}\,[w]_{A_{p,p*}}^{\frac{1}{n'}\max\{1,\frac{p'}{p*}\}} \left(\int_Q |w\nabla f|^p dx\right)^{\frac{1}{p}}. \tag{2.6}$$

However (2.6) has being improved in Corollary 1.4 combining (1.2) of our measured version of the Isoperimetric inequality Theorem 1.1 with a variation of the following update version of the HMS extrapolation theorem which can be found in [25].

Before stating the theorem we need a general version of the $A_{p,p*}$ class already introduced. In [28], the authors characterized the weighted strong-type inequality for fractional operators in terms of the so-called $A_{p,q}$ condition. For $1 < p < \frac{n}{\alpha}$ and q defined by

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},$$

they showed that for a finite constant C and for all $f \geqslant 0$,

$$\|wT_\alpha f\|_{L^q(\mathbb{R}^n)} \leqslant C\,\|wf\|_{L^p(\mathbb{R}^n)}$$

where $T_\alpha = I_\alpha$ or M_α, if and only if $w \in A_{p,q}$, namely

$$[w]_{A_{p,q}} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w^q\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\,dx\right)^{q/p'} < \infty.$$

The sharp version of the HMS theorem is the following.

Theorem 2.1. *Suppose that T is an operator defined on an appropriate class of functions. Suppose further that p_0 and q_0 are exponents with $1 \leqslant p_0 \leqslant q_0 < \infty$, and such that*

$$\|wTf\|_{L^{q_0}(\mathbb{R}^n)} \leqslant c[w]^{\gamma}_{A_{p_0,q_0}}\|wf\|_{L^{p_0}(\mathbb{R}^n)}$$

holds for all $w \in A_{p_0,q_0}$ and some $\gamma > 0$. Then,

$$\|wTf\|_{L^q(\mathbb{R}^n)} \leqslant c[w]_{A_{p,q}}^{\gamma\max\{1,\frac{q_0}{p_0'}\frac{p'}{q'}\}}\|wf\|_{L^p(\mathbb{R}^n)}$$

holds for all p and q satisfying $1 < p \leqslant q < \infty$ and

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0},$$

and all weight $w \in A_{p,q}$.

Two important remarks are in order. The first one is that the proof provided in [25] follows the original one in [21], except for the careful track of the dependence of the estimates in terms of the $A_{p,q}$ constants. The second remark is the observation that there is no need to consider operators T as such, it is enough to consider "inequalities with weights", which are well defined on each step. In our context we are dealing with Poincaré or Poincaré–Sobolev type estimates with the appropriate weights. An idea of this sort was considered in [32], in the proof of Corollary 1.11.

3. Proofs

For the proofs we will use the following lemma. We use here the standard notation, for $0 < \alpha < n$, we define the Riesz potential of a non-negative measurable function u by

$$I_\alpha u(x) = \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}}\,dy\,, \qquad x \in \mathbb{R}^n \tag{3.1}$$

Lemma 3.1. Let Q_0 be a cube in $\mathbb{R}^n$, μ be a measure and $0 < \alpha < n$. Then

$$I_\alpha(\chi_{Q_0}\mu)(x) \leqslant \frac{c_n}{\alpha}\mu(Q_0)^{\frac{\alpha}{n}} M^c(\mu\chi_{Q_0})(x)^{\frac{n-\alpha}{n}} \tag{3.2}$$

for a.e. $x \in Q_0$. In particular we have

$$I_\alpha(\chi_{Q_0}\mu)(x) \leqslant \frac{c_n}{\alpha}\ell(Q_0)^\alpha\, M^c(\mu\chi_{Q_0})(x) \tag{3.3}$$

and if $d\mu(y) = w(y)\,dy$ for some $w \in A_1$, then

$$I_\alpha(\chi_{Q_0}\mu)(x) \leqslant \frac{c_n}{\alpha}[w]_{A_1}^{\frac{n-\alpha}{n}}\, w(Q_0)^{\frac{\alpha}{n}}\, w(x)^{\frac{n-\alpha}{n}} \tag{3.4}$$

for a.e. $x \in Q_0$.

Estimate (3.2) is probably known, but we will provide an unusual argument, at least in this context, for the convenience of the reader, which can also be found in a work in preparation by I. Gardeazabal, E. Loriest, and C. Pérez. On the other hand, (3.3) is well known.

Proof. For $t > 0$ we let

$$Q_{x,t} := Q\left(x, t^{-\frac{1}{n-\alpha}}\right)$$

be the cube with centre at x and sidelength $t^{-\frac{1}{n-\alpha}}$. Then, using the layer-cake formula, we obtain

$$\begin{aligned}
\int_{Q_0} \frac{d\mu(y)}{|x-y|^{n-\alpha}} &= \int_0^\infty \mu\left(\left\{y \in Q_0 : \frac{1}{|x-y|^{n-\alpha}} > t\right\}\right) dt \\
&= \int_0^\infty \mu\left(\left\{y \in Q_0 : |x-y| < t^{-\frac{1}{n-\alpha}}\right\}\right) dt \\
&\leqslant \int_0^\infty \min\left\{\mu(Q_0), \frac{\mu(Q_{x,t})}{|Q_{x,t}|}\,|Q_{x,t}|\right\} dt. \\
&\leqslant c_n \int_0^\infty \min\left\{\mu(Q), M^c(\mu)(x)\left(t^{-\frac{1}{n-\alpha}}\right)^n\right\} dt \\
&= c_n \int_0^{\left(\frac{M^c(\mu)(x)}{\mu(Q)}\right)^{\frac{n-\alpha}{n}}} \mu(Q)\, dt \\
&\quad + c_n \int_{\left(\frac{M^c(\mu)(x)}{\mu(Q)}\right)^{\frac{n-\alpha}{n}}}^\infty M^c(\mu)(x) t^{-\frac{n}{n-\alpha}}\, dt \\
&= \frac{c_n}{\alpha}\mu(Q)^{\frac{\alpha}{n}} M^c(\mu)(x)^{\frac{n-\alpha}{n}}.
\end{aligned}$$

From here, (3.3) and (3.4) are deduced immediately. □

Proof of Theorem 1.1 . Using the pointwise estimate from (2.3)

$$|f(x) - f_Q| \leqslant c_n\, I_1(|\nabla f|\chi_Q)(x),$$

it is enough to prove that

$$\|I_1(|\nabla f|\chi_Q)|_{L^{n',\infty}(Q,d\mu)} \leqslant C \int_Q |\nabla f(x)| M^c(\chi_Q\mu)(x)^{\frac{1}{n'}}\, dx. \tag{3.5}$$

Consider the set defined by

$$E_Q := \{x \in Q : I_1(|\nabla f|\chi_Q)(x) > 1\}$$

Then, since I_1 is self-adjoint,

$$\mu(E_Q) \leqslant \int_{E_Q} I_1(|\nabla f|\chi_Q)(x)d\mu(x) = \int_Q |\nabla f(x)|\, I_1(\chi_{E_Q}\mu)(x)\,dx$$

Using Lemma 3.1 with $\nu = \chi_{E_Q}\mu$ we obtain

$$\mu(E_Q) \leqslant c_n\, \mu(E_Q)^{\frac{1}{n}} \int_Q |\nabla f(x)|\, M^c(\chi_Q\mu)(x)^{\frac{1}{n'}}\,dx.$$

Hence, assuming $\mu(E_Q) > 0$ (otherwise the inequality would be trivial)

$$\mu(E_Q)^{\frac{1}{n'}} \leqslant c_n \int_Q |\nabla f(x)|\, M^c(\chi_Q\mu)(x)^{\frac{1}{n'}}\,dx.$$

Since this inequality is homogeneous in f we can replace f by $\frac{f}{t}$, $t > 0$. Then for any $t > 0$,

$$t\mu\{x \in Q : I_1(|\nabla f|)(x) > t\}^{\frac{1}{n'}} \leqslant c_n \int_Q |\nabla f(x)|\, M^c(\chi_Q\mu)(x)^{\frac{1}{n'}}\,dx.$$

This yields (3.5) and hence the claim (1.1). □

We now proceed with the proof of the corollaries.

Proof of Corollary 1.2 . We will use the approach from [35] via mollified characteristic functions. More precisely, consider $u = \chi_\Omega$ and, for any $\varepsilon > 0$, the convolution

$$u_\varepsilon(x) := (J_\varepsilon * u)(x),$$

where $J_\varepsilon(x) = \frac{1}{\varepsilon^n} J(x/\varepsilon)$ for a good radial decreasing kernel J normalized so it has integral 1 over $\mathbb{R}^n$. We use (1.3) with u_ε instead of f and take the limit when $\varepsilon \to 0$. The left hand side goes to $\mu(\Omega)^{\frac{1}{n'}}$ by using easy to check properties of approximate identities. For the left hand side, we use a very nice connection between convoluted radial decreasing kernels and maximal functions. More precisely, we know that under the hypothesis on J, we have

$$\sup_{\varepsilon>0} |(J_\varepsilon * f)(x)| \leqslant Mf(x),$$

since the L^1 norm of J is one. In our situation, we use Gauss-Green equations to obtain

$$|\nabla u_\varepsilon(x)| \leqslant \int_{\partial\Omega} J_\varepsilon(x-y)\; d\mathcal{H}^{n-1}(y). \tag{3.6}$$

Integrating on the x variable as in (1.3) we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u_\varepsilon| \,(M^c\mu)^{\frac{1}{n'}}\ dx &\leqslant \int_{\mathbb{R}^n} (M^c\mu)^{\frac{1}{n'}}(x) \int_{\partial\Omega} J_\varepsilon(x-y)\ d\mathcal{H}^{n-1}(y)\ dx\\ &\leqslant \int_{\partial\Omega}\int_{\mathbb{R}^n} (M^c\mu)^{\frac{1}{n'}}(x) J_\varepsilon(x-y)\ dx\ d\mathcal{H}^{n-1}(y)\\ &= \int_{\partial\Omega} (J_\varepsilon * (M^c\mu)^{\frac{1}{n'}})(y) d\mathcal{H}^{n-1}(y)\\ &\leqslant \int_{\partial\Omega} M((M^c\mu)^{\frac{1}{n'}}))(y)\mathcal{H}^{n-1}(y).\end{aligned}$$

We used (3.6) in the last inequality. We conclude by observing that $(M^c\mu)^{\frac{1}{n'}})$ is in fact an A_1 weight and not only that, we also know how to compute its A_1 constant(see (1.4)):

$$\left[(M^c\mu)^{\frac{1}{n'}})\right]_{A_1} \leqslant \frac{c_n}{1-\frac{1}{n'}} = c_n n.$$

We obtain then the uniform control

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon(x)| \,(M^c\mu(x))^{\frac{1}{n'}}\ dx \leqslant c_n \int_{\partial\Omega} (M^c\mu)^{\frac{1}{n'}})(y)\mathcal{H}^{n-1}(y)$$

for every positive ε. Taking the limit when $\varepsilon \to 0$, we get the claimed inequality

$$\mu(\Omega)^{\frac{n-1}{n}} \leqslant c_n \int_{\partial\Omega} (M^c\mu)^{\frac{1}{n'}}\ d\mathcal{H}^{n-1}.$$

□

Proof of Corollary 1.4 . For a fixed Q, we begin with estimate (1.2),

$$\|f-f_Q\|_{L^{n'}(Q,d\mu)} \leqslant c_n \int_Q |\nabla f(x)|(M^c(\chi_Q\mu)(x))^{\frac{1}{n'}}\ dx,$$

If $d\mu = w^{n'}dx$ with $w \in A_{1,n'}$, we have

$$\|w(f-f_Q)\|_{L^{n'}(Q,dx)} \leqslant c_n\,[w]_{1,n'}^{\frac{1}{n'}} \int_Q |\nabla f(x)|\ w(x)\,dx,$$

We now apply the HMS extrapolation Theorem 2.1 whose proof can be adapted easily to this *non-operator* context with a gradient on the right hand side,[1] with parameters $p_0 = 1$, $q_0 = n'$, $\gamma = \frac{1}{n'}$, to get

$$\|w(f-f_Q)\|_{L^{p*}(Q,dx)} \leqslant c_{n,p}\,[w]_{1,n'}^{\frac{1}{n'}} \left(\int_Q |w\nabla f|^p dx\right)^{\frac{1}{p}}.$$

Observe an important point: the exponent $\max\{1, \frac{q_0}{p_0'}\frac{p'}{q'}\}$ in the outcome of the Theorem 2.1 equals one since $p_0 = 1$ in our case. □

[1] We remark that it is not clear how to do it in the fractional context. This is the main obstacle to prove Conjecture 1.9.

For the proof of Theorem 1.7 we will need the following precise representation lemma used in [22] based on the ideas in [14, 15]. The notation we use has been already defined in (3.1).

Lemma 3.2. Let Q_0 be a cube in $\mathbb{R}^n$. Assume that $0 < \alpha < n$ and consider $0 < \eta < n - \alpha$. Let $f \in L^1(Q_0)$ and let g be a non-negative measurable function on Q_0 such that for a finite constant κ,

$$\fint_Q |f(x) - f_Q|\, dx \leqslant \kappa\, \ell(Q)^\alpha \fint_Q g(x)\, dx$$

for every cube $Q \subset Q_0$. Then there exists a dimensional constant c_n such that

$$|f(x) - f_{Q_0}| \leqslant c_n \frac{\kappa}{\eta} I_\alpha(g\chi_{Q_0})(x)$$

for almost every $x \in Q_0$. In the particular case that $\alpha = \eta < \frac{n}{2}$,

$$|f(x) - f_{Q_0}| \leqslant c_n \frac{\kappa}{\alpha}\, I_\alpha(g\chi_{Q_0})(x).$$

Proof of Theorem 1.7. We will make use of the following "initial" starting point

$$\fint_Q |f(x) - f_Q|\, dx \leqslant c_n\, (1-\delta)\ell(Q)^\delta \fint_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}}\, dy\, dx,$$

which can be found in [2]. Then apply Lemma 3.2 with $\alpha = \delta$, $\kappa = c_n\,(1-\delta)$ and

$$g_{f,Q}(x) := \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}}\, dy\, \chi_Q(x).^2$$

Then there exists a constant $c = c(n)$ such that

$$|f(x) - f_Q| \leqslant c_n\,(1-\delta)\, I_\delta(\chi_Q g_{f,Q})(x) \tag{3.7}$$

for almost every $x \in Q$.

We claim first a corresponding weak version of (1.9) from Theorem (1.7), namely

$$\|f - f_Q\|_{L^{\frac{n}{n-\delta},\infty}(Q,d\mu)} \leqslant c_n\,(1-\delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}}\, dy\, M\mu(x)^{\frac{n-\delta}{n}}\, dx. \tag{3.8}$$

By (3.7) it is enough to prove

$$\|I_\delta(\chi_Q g_{f,Q})\|_{L^{\frac{n}{n-\delta},\infty}(Q,d\mu)} \leqslant c_n \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}}\, dy\, M\mu(x)^{\frac{n-\delta}{n}}\, dx. \tag{3.9}$$

Define,

$$E_Q := \{x \in Q : I_\delta(\chi_Q g_{f,Q})(x) > 1\}$$

Since I_δ is self-adjoint,

$$\begin{aligned}\mu(E_Q) &\leqslant \int_{E_Q} I_\delta(\chi_Q g_f)(x) d\mu(x) = \int_Q g_f(x)\, I_\delta(\chi_{E_Q}\mu)(x)\, dx \\ &\leqslant c_n \mu(E_Q)^{\frac{\delta}{n}} \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}}\, (M\mu(x))^{\frac{n-\delta}{n}}\, dx\end{aligned}$$

[2] The proof of Lemma 3.2 works for this very special case as well.

by Lemma 3.1 and hence, assuming $\mu(E_Q) > 0$,

$$\mu(E_Q)^{1-\frac{\delta}{n}} \leqslant c_n \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} \left(M\mu(x)\right)^{\frac{n-\delta}{n}} \, dx.$$

By homogeneity, we can replace f by $\frac{f}{t}$ with $t > 0$ which yields the estimate (3.9) which proves the claim (3.8).

To finish the proof of the theorem we use the "truncation method" that works as well in this context. We remit to [22] for a full account. □

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