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AVATARS OF STEIN'S THEOREM IN THE COMPLEX SETTING

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This paper is dedicated to Eleonor Harboure, Pola, a colleague and friend whose memory will live on

Abstract. In this paper, we establish some variants of Stein's theorem, which states that a non-negative function belongs to the Hardy space $H^1(\mathbb{T})$ if and only if it belongs to $L\log L(\mathbb{T})$. We consider Bergman spaces of holomorphic functions in the upper half plane and develop avatars of Stein's theorem and relative results in this context. We are led to consider weighted Bergman spaces and Bergman spaces of Musielak–Orlicz type. Eventually, we characterize bounded Hankel operators on $A^1(\mathbb{C}_+)$.

1. Introduction

This article follows naturally our previous work [4] where we considered an analog of Stein's Theorem, but with an estimate on the whole space $\mathbb{R}^n$ instead of a local estimate. Recall that Stein's Theorem ([12]) says that, whenever f is an L^1 non negative function, which is supported in a ball B, its Hardy–Littlewood maximal function Mf is integrable on B if and only if $f\ln(e+f)$ is integrable. When working on the whole $\mathbb{R}^n$, one may be interested in having f in the Hardy space $H^1(\mathbb{R}^n)$, that is, the space of integrable functions such that all its Riesz transforms are integrable. In this setting, the fact that a function of $H^1(\mathbb{R}^n)$ has integral 0 induces to replace the assumption f non negative by $f = g - (\int g)\chi_Q$ with g non negative and Q a fixed cube of measure 1. We then proved in [4] that f is in $H^1(\mathbb{R}^n)$ if and only if $|f|(\ln(e+|f|)+\ln(e+|x|))$ is integrable. This condition may be interpreted as the fact that f belongs to a Musielak-Orlicz space, related to the Musielak function $(x,t)\mapsto t(\ln(e+t)+\ln(e+|x|))$.

It was natural to consider the same problem in the upper half-space

$$\mathbb{C}_+ := \{z\in\mathbb{C},\ \Im m(z)>0\},$$

with the Bergman projection P replacing the Riesz transforms. It happens that in this case we do not need Musielak-Orlicz spaces, but only weighted L^1 spaces. Here $L^1(\mathbb{C}_+)$ is the space of integrable functions for the volume measure on $\mathbb{C}_+$

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noted as

$$dV(z) = dx\,dy \qquad z = x + iy,$$

while $L^1_\Omega(\mathbb{C}_+)$ is the space of integrable function for the measure ΩdV. As usual, the weight Ω is assumed to be measurable and non negative. Recall that the kernel of the Bergman projection P is given by

$$K(z,\zeta) = \frac{1}{\pi(z-\overline{\zeta})^2}.$$

We call P^+ the operator with kernel $|K(z,\zeta)|$.

Our analog of Stein's theorem is the following.

Theorem 1. *Assume that f is an integrable function, which is supported in a strip $\{|x|<a\}$. Then Pf is integrable in any strip $\{|x|<b\}$ if $f\ln(e+y^{-1})$ is integrable. Moreover, when f is also non negative, this is a necessary and sufficient condition for P^+f to be integrable in the strip $\{|x|<a\}$.*

Our next aim is to suppress the assumption on the support of f as in the case of $H^1(\mathbb{R}^n)$. Let us recall that integrable holomorphic functions have integral zero. Moreover, as we will see, a function $f \in L^1 \cap L^2$ such that Pf is in L^1, has itself integral zero. This leads us to subtract a term to f, as we did in $\mathbb{R}^n$, when f does not have integral zero.

So we consider the operator T_K, defined by

$$T_K f(z) = \int_{\mathbb{C}_+} K(z,\zeta) f(\zeta) dV(\zeta) - \left(\int_{\mathbb{C}_+} f(\zeta)\, dV(\zeta)\right) K(z,i).$$

We prove that T_K maps the weighted space $L^1(\omega dV)$ into L^1 with

$$\omega(z) := \ln\left(e + \frac{1}{\Im m(z)}\right) + \ln(e+|z|), \quad z \in \mathbb{C}_+. \tag{1.1}$$

Moreover there is a kind of converse when the Bergman kernel is replaced by its modulus.

As in the case of $\mathbb{R}^n$, the condition that appears here is reminiscent of other properties that involve Bergman spaces, and which we also give in this paper. Indeed, while the Musielak-Orlicz function given above appeared for products of a function in $H^1(\mathbb{R}^n)$ and a function of its dual, this weight ω appears for products of a holomorphic function of the Bergman space $A^1(\mathbb{C}_+)$ and a function of its dual space.

Let us fix some notations before going on. We use the notation $\mathcal{H}(\mathbb{C}_+)$ for the set of holomorphic functions in $\mathbb{C}_+$. We recall that for $0<p<\infty$, the Hardy space $H^p(\mathbb{C}_+)$ consists of all analytic functions f on $\mathbb{C}_+$ such that

$$\|f\|_p := \sup_{y>0}\left(\int_{\mathbb{R}} |f(x+iy)|^p dx\right)^{1/p} < \infty.$$

Let $1 \leq p < \infty$. For a weight Ω defined on $\mathbb{C}_+$, we denote by $L^p_\Omega(\mathbb{C}_+)$, the set of all measurable functions f on $\mathbb{C}_+$ such that

$$\|f\|^p_\Omega := \int_{\mathbb{C}_+} |f(z)|^p \Omega(z)\, dV(z) < \infty.$$

The Bergman space $A^p_\Omega(\mathbb{C}_+)$ is the subset of $L^p_\Omega(\mathbb{C}_+)$ consisting of holomorphic functions. We remark that depending on the weight Ω, this set can be trivial.

When $\Omega(z) = (\Im m(z))^\alpha$, we recover the classical (weighted) Bergman space denoted by $A^p_\alpha(\mathbb{C}_+)$ (see [2]). This space is nontrivial only if $\alpha > -1$.

For $\alpha > -1$, the Bergman projection P_α is the orthogonal projection from $L^2_\alpha(\mathbb{C}_+)$ into its closed subspace $A^2_\alpha(\mathbb{C}_+)$. It is defined by

$$P_\alpha f(z) = \int_{\mathbb{C}_+} K_\alpha(z,w) f(w) dV_\alpha(w)$$

where

$$K_\alpha(z,w) = \frac{c_\alpha}{(z-\bar{w})^{2+\alpha}}$$

is the Bergman kernel, $dV_\alpha(w) = (\Im m(w))^\alpha dV(w)$ and c_α is a constant that depends only on α.

Our proofs will use different values of α, and our results can be generalized to all $\alpha > -1$, but we come back in this introduction to the particular case of $\alpha = 0$. It is well-known (see [8]) that the dual space of $A^1(\mathbb{C}_+)$ is the space of Bloch functions modulo constants. Recall that a function f is a Bloch function if

$$\Im m(z)|f'(z)| \leq C.$$

It is easy to see that the product of a function $f \in A^1(\mathbb{C}_+)$ and a Bloch function g belongs to the space $L^1_{\omega^{-1}}(\mathbb{C}_+)$, see Corollary 1. In fact, as we shall see in §6 below, it belongs to the a priori smaller space $A_{\log}(\mathbb{C}_+)$ of holomorphic functions such that $f/(\ln(e+|f|)+\ln(e+|z|)$ is integrable. It can be seen that $A^1_{\omega^{-1}}(\mathbb{C}_+)$ is the smallest Banach space containing $A_{\log}(\mathbb{C}_+)$. In particular, they have the same dual space, and the multipliers of the Bloch space are characterized in terms of this dual space, that is, they belong to the space $\mathcal{B}_{\log}(\mathbb{C}_+)$ of holomorphic functions f such that

$$\Im m(z)|f'(z)| \leq \frac{C}{\omega(z)}.$$

Finally, one would like to know whether a function in $A^1_{\omega^{-1}}(\mathbb{C}_+)$ is weakly factorized, that is, may be written as a sum $\sum f_j g_j$ with $f_j \in A^1(\mathbb{C}_+)$ and g_j Bloch functions, with an adapted normalization. As in the seminal paper of Coifman and Rochberg [7], this is done after an atomic decomposition of the space, based on the fact that the projector P_α, $\alpha > 0$ reproduces the functions of $A^1_{\omega^{-1}}(\mathbb{C}_+)$. As a consequence, we characterize the symbols of bounded Hankel operators (see Paragraph 5.3 for the definition).

Theorem 2. *Let b be in $\mathcal{B}(\mathbb{C}_+)$. Then the Hankel operator h_b extends to a bounded operator from $A^1(\mathbb{C}_+)$ to itself if and only if $b \in \mathcal{B}_{log}(\mathbb{C}_+)$.*

It should be emphasized that all this can be adapted to the unit disk and, as well to the unit ball of $\mathbb{C}^n$. One can consult [3] on factorization properties. Our aim, here, is to understand how the behavior at infinity influences these questions.

When the Bergman projection is replaced by the Szegö projection and Bergman spaces by Hardy spaces, part of the results are contained in our previous work [4]. The factorization questions are treated in [6] and involve a Hardy space of Musielak type. Remark that, in this case, one has an exact factorization. We do not know whether one can replace weak factorization by exact factorization in the Bergman setting. There is a large literature on products of functions respectively in H^1 and BMO (see in particular [5, 9]).

The paper is organized as follows. In Section 2, we start by studying the integrability of the modulus of the Bergman kernels with respect to the weights ω and ω^{-1}. Such integrability properties are fundamental throughout the paper. In Section 3, we prove Theorem 1 and extend it to non-compactly supported functions. Then, we consider L^∞-problems through duality and Bloch spaces in Section 4. By the way, we characterize the multipliers of Bloch space and the dual space of $A^1_{\omega^{-1}}(\mathbb{C}_+)$. Section 5 is devoted to products of functions in $A^1(\mathbb{C}_+)$ and Bloch and to Hankel operators. The proof of Theorem 2 is given. We end the paper by proving that $A^1_{\omega^{-1}}(\mathbb{C}_+)$ is the smallest Banach spaces containing the products of functions in $A^1(\mathbb{C}_+)$ and Bloch.

In the following, we will use the notation $A \lesssim B$ (respectively $A \gtrsim B$, $A \simeq B$) whenever there exists a uniform constant $C > 0$ such that $A \leq CB$ (respectively $A \geq CB$, respectively $A \lesssim B$ et $A \gtrsim B$).

2. Weights and reproducing formulas

In the following we will consider the weight ω, already defined (1.1) in the introduction by

$$\omega(z) := \ln\left(e + \frac{1}{\Im m(z)}\right) + \ln(e + |z|), \quad z \in \mathbb{C}_+.$$

We establish the following lemma.

Lemma 1. *Let $\alpha > 0$, and $\beta > -1$. Then there is a constant $C = C_{\alpha,\beta} > 0$ such that for any $z_0 \in \mathbb{C}_+$,*

$$\int_{\mathbb{C}_+} \frac{\omega(z)\, dV_\beta(z)}{|z - \overline{z}_0|^{2+\alpha+\beta}} \leq C(\Im m(z_0))^{-\alpha}\omega(z_0).$$

$$\int_{\mathbb{C}_+} \frac{dV_\beta(z)}{|z - \overline{z}_0|^{2+\alpha+\beta}\omega(z)} \leq C\frac{(\Im m(z_0))^{-\alpha}}{\omega(z_0)}.$$

Proof. Let I be the interval centered at $\Re e z_0$ with length $2\Im m(z_0)$. We denote by Q_{z_0} the square over I. Remark that there exist two constants c and C such that

$$B(\overline{z}_0, c\Im m(z_0)) \cap \mathbb{C}_+ \subset Q_{z_0} \subset B(\overline{z}_0, C\Im m(z_0)). \tag{2.1}$$

Here the notation $B(\cdot,\cdot)$ stands for an Euclidean ball in the complex plane.

We will only consider the second inequality of the lemma. The first one follows from slight modifications. We start by proving that

$$\int_{Q_{z_0}} \frac{dV_\beta(z)}{\omega(z)} \lesssim \frac{(\Im m(z_0))^{2+\beta}}{\omega(z_0)}. \tag{2.2}$$

Remark 1. *Inequality* (2.2) *may be interpreted as the fact that ω^{-1} belong to the Bèkollè-Bonami class B_1 of weights (see* [1]*). The same holds for ω. This is a necessary and sufficient condition for having weak L^1 inequalities for the Bergman projection.*

Recall that Q_{z_0} satisfies (2.1). We split the proof of (2.2) into three cases: $\omega(z_0) \simeq 1$, $\omega(z_0) \simeq \ln(1/y_0)$ and $\omega(z_0) \simeq \ln(|z_0|)$. Here, we write $z_0 = x_0 + iy_0$ and $z = x + iy$. The proof is direct when $\omega(z_0) \simeq 1$. Let us next assume that $y_0 < 1/2$ and $\omega(z_0) \simeq \ln(1/y_0)$. In this case, it is sufficient to prove that

$$\int_{Q_{z_0}} \frac{y^\beta}{\ln(1/y)} dxdy \lesssim \frac{y_0^{2+\beta}}{\ln(e + 1/y_0)},$$

which is straightforward. Assume finally that $\omega(z_0) \simeq \ln(|z_0|)$, with $|z_0| > 2$. If $|z_0| > 2y_0$, then $|z| \simeq |z_0|$ and $\omega(z) \gtrsim \ln(|z_0|)$, so that we conclude directly. If $2y_0 > |z_0| > 2$, then $\omega(z_0) \simeq \ln(y_0)$. We write now that

$$\int_{Q_{z_0}} \frac{y^\beta}{\ln(y)} dxdy \lesssim \frac{y_0^{2+\beta}}{\ln(y_0)}.$$

We have proved (2.2) in all cases.

Let us go on with the proof. For $j \in \mathbb{N}$, we define $z_j := x_0 + i2^j y_0$. Put $E_0 = Q_{z_0}$ and for $j \geq 1$, $E_j = Q_{z_j} \setminus Q_{z_{j-1}}$. The inequality (2.2) is valid for z_j in place of z_0, so that

$$\int_{Q_{z_j}} \frac{dV_\beta(z)}{\omega(z)} \lesssim \frac{2^{(2+\beta)j}(\Im m(z_0))^{2+\beta}}{\omega(z_j)}.$$

Moreover, for $j \geq 1$, the fact that z does not belong to $Q_{z_{j-1}}$ implies that $|z - \overline{z_0}| \geq c2^{j-1}\Im m(z_0)$. So, for $j \geq 1$,

$$\int_{E_j} \frac{dV_\beta(z)}{|z - \overline{z_0}|^{2+\alpha+\beta}\omega(z)} \lesssim \frac{2^{-\alpha j}(\Im m(z_0))^{-\alpha}}{\omega(z_j)}.$$

A computation analogous to the one we did for (2.2) gives the same estimate for $j = 0$. As $\mathbb{C}_+$ is the union of the sets $E_j, j \geq 0$, it remains to prove that

$$\sum_{j=0}^{\infty} 2^{-\alpha j}\omega(z_j)^{-1} \lesssim \omega(z_0)^{-1}.$$

From the expression of ω, one deduces that $\omega(z_j) \geq \omega(z_0) - j\ln 2$. We also know that $\omega \geq 1$. We cut the sum into two parts, the first one with $j < \omega(z_0)/2$. Both sums are bounded by a constant times $\omega(z_0)^{-1}$, which we wanted to have. □

Remark 2. *The same proof allows us to obtain the same conclusions as in Lemma 1 when ω is replaced on both sides by one of the three weights ω_j, $j = 0, 1, 2$, with $\omega_0 \equiv 1$, $\omega_1(z) = \ln(e + (\Im m(z))^{-1})$ and $\omega_2(z) = \ln(e + |z|)$.*

Remark 3. *We have as well bounds below in Lemma 1: just write that the integral is bounded below by the integral on the disc with center z_0 and radius $\Im m(z_0)/2$. Within this disc, which has measure $c\Im m(z_0)^{2+\beta}$, the function to integrate is equivalent to $\Im m(z_0)^{-2-\alpha-\beta}\omega(z_0)^{\pm 1}$.*

Let us use the notation $L^1_{\omega^{-1},\alpha}(\mathbb{C}_+) = L^1(\mathbb{C}_+, \omega^{-1}(z)\, dV_\alpha(z))$.

Proposition 1. *Let $\alpha > 0$, and $\beta > -1$. Then the Bergman projector $P_{\alpha+\beta}$ is bounded from $L^1_{\omega^{-1},\beta}(\mathbb{C}_+)$ to $L^1_{\omega^{-1},\beta}(\mathbb{C}_+)$.*

Proof. This is a direct consequence of Fubini Theorem and Lemma 1. □

Let us now introduce the following weighted space.

$$L^\infty_{\omega,\alpha}(\mathbb{C}_+) := \{f \text{ measurable} : \sup_{z\in\mathbb{C}_+} (\Im m(z))^\alpha \omega(z)|f(z)| < \infty\}.$$

Then $L^\infty_{\omega,\alpha}(\mathbb{C}_+)$ is a Banach space with norm

$$\|f\|_{\infty,\omega,\alpha} := \sup_{z\in\mathbb{C}_+} (\Im m(z))^\alpha \omega(z)|f(z)|.$$

Proposition 2. *Let $\alpha > 0$. Then the Bergman projection $P_{2\alpha}$ maps $L^\infty_{\omega,\alpha}(\mathbb{C}_+)$ boundedly into itself.*

Proof. That $P_{2\alpha}$ is well-defined and bounded on $L^\infty_{\omega,\alpha}(\mathbb{C}_+)$ follows directly from Lemma 1. □

3. Avatars of Stein theorem in the Bergman setting

3.1. **Local smoothness.** This subsection is devoted to the proof of Theorem 1. We first prove the sufficient condition. It is sufficient to consider P^+, since $|Pf| \leq P^+f$. We use the notations $z = x + iy$ and $\zeta = u + iv$. The integrability of P^+f for $y > a$ follows at once. Indeed, $|z - \overline{\zeta}|^{-2} < y^{-2}$, and the integral of $y^{-2}\chi_{y>a}$ on the strip $|x| < b$ is bounded by $\frac{2b}{a}$. It remains to consider the integral of $|z - \overline{\zeta}|^{-2}$ on the domain $|x| < a, y < a$. One has

$$\begin{aligned}\int_{|x|<a,y<a} |z - \overline{\zeta}|^{-2} dV(z) &\leq \int_{|x|>0,y<a} \left(x^2 + (y+v)^2\right)^{-1} dx\, dy \\ &= \pi \int_{0<y<a} \frac{dy}{y+v} = \pi \ln(1 + a/v)\end{aligned}$$

which allows us to conclude.
The necessary condition for P^+ comes from the estimate

$$\int_{|x|<a,y<a/2} P_+f(x+iy)\,dxdy \geq \int_{|u|<a,v<a/2} f(u+iv)J(u,v)dudv,$$

where, for $|u|<a$ and $v<a/2$,

$$\begin{aligned} J(u,v) &= \int_{|x|<a,y<a/2} \left((x-u)^2+(y+v)^2\right)^{-1} dx\,dy \\ &\geq \int_{|x-u|<y+v,|x|<a,y<a/2} \frac{dxdy}{(y+v)^2} \\ &\geq \int_0^{a/2} \frac{dy}{y+v} = \frac{1}{2}\ln\left(1+\frac{a}{2v}\right). \end{aligned}$$

We have used the fact that the interval of x's such that $|x|<a, |x-u|<y+v$ has length at least $y+v$. The conclusion follows at once.

3.2. **Smoothness of L^1-type.** In this section, we are interested in sufficient conditions which ensure that a holomorphic function is integrable on $\mathbb{C}_+$. We first prove the next lemma.

Lemma 2. *We have the inequality*

$$\int_{\mathbb{C}_+} \left|(z-\overline{\zeta})^{-2}-(z+i)^{-2}\right| dV(z) \lesssim \omega(\zeta).$$

Proof. We write $z=x+iy$ and $\zeta=\xi+i\lambda$. We cut the domain of integration into pieces.

(1) $y<1$: we consider the two terms separately: the second one has a bounded integral, while integrating first in x, we find that

$$\int_{y<1}\int_{\mathbb{R}} |z-\overline{\zeta}|^{-2}dx\,dy = \pi\int_{y<1}(y+\lambda)^{-1}dy = \pi\ln(1+\frac{1}{\lambda}) \lesssim \ln(e+\frac{1}{\lambda}).$$

(2) $|\zeta-i|<|z+i|/2$: the function to integrate is bounded by $C\frac{|\zeta-i|}{|z+i|^3}$, whose integral is bounded by a constant in this domain of integration.

(3) $y>1$ and $|z+i|\leq 2|\zeta-i|$: we again consider separately the two terms and start by integrating in x. We get twice the same bound

$$\int_1^{2|\zeta|+2}\frac{dy}{y} \lesssim \ln(e+|\zeta|).$$

□

Remark 4. *As a consequence, the following inequality holds:*

$$\left|\int_{\mathbb{C}_+}\left(|z-\overline{\zeta}|^{-2}-|z+i|^{-2}\right) dV(z)\right| \lesssim \omega(\zeta).$$

Define the following operator for $f \in L^1(\mathbb{C}_+)$ and $z \in \mathbb{C}_+$

$$T_K f(z) := Pf(z) - \Big(\int f dV\Big) K(z, i).$$

Here K stands for the Bergman kernel K_0. Remark that for functions of integral 0, this operator coincides with the Bergman projector.

As an immediate consequence of the lemma, we get the following result.

Theorem 3. *The operator T_K maps $L^1(\omega dV)$ into $A^1(\mathbb{C}_+)$.*

The proof is direct since the kernel of T_K is given by

$$(z, \zeta) \mapsto \frac{1}{(z - \overline{\zeta})^2} - \frac{1}{(z + i)^2}.$$

Remark 5. *As functions in $A^1(\mathbb{C}_+)$ have integral 0, for a function f in $A^1(\mathbb{C}_+)$ we have*

$$T_K f = Pf = f.$$

Remark 6. *As stated in the introduction, if $f \in L^1(\mathbb{C}_+) \cap L^2(\mathbb{C}_+)$ and $Pf \in L^1(\mathbb{C}_+)$, then f also has integral 0 so that $T_K f = Pf$. Indeed, since $Pf \in A^1(\mathbb{C}_+)$, it has integral zero so that, for $m > 4$,*

$$\begin{aligned}
0 &= \int_{\mathbb{C}_+} Pf(z) dV(z) \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{C}_+} Pf(z)(1 + i\epsilon \overline{z})^{-m} dV(z) \\
&= \lim_{\epsilon \to 0} \langle Pf, (1 - i\epsilon z)^{-m} \rangle = \lim_{\epsilon \to 0} \langle f, P(1 - i\epsilon z)^{-m} \rangle \\
&= \lim_{\epsilon \to 0} \langle f, (1 - i\epsilon z)^{-m} \rangle \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{C}_+} f(\zeta)(1 + i\epsilon \overline{\zeta})^{-m} dV(\zeta) \\
&= \int_{\mathbb{C}_+} f(\zeta) dV(\zeta).
\end{aligned}$$

From Remark 4, the operator of kernel

$$(z, \zeta) \mapsto \frac{1}{|z - \overline{\zeta}|^2} - \frac{1}{|z + i|^2}$$

has the same boundedness property. We are going to prove that it also satisfies a partial converse result which may be interpreted as an avatar of Stein's Theorem in the Bergman setting.

Theorem 4. *The operator $|T_K|$ defined by*

$$|T_K|f := \int |K(\cdot,\zeta)|f(\zeta)\,dV(\zeta) - \int f dV \times |K(\cdot, i)|$$

maps $L^1(\omega dV)$ into $L^1(dV)$. Moreover, for positive integrable functions f, the integrability condition $f \in L^1(\omega dV)$ is necessary to get $|T_K|f \in L^1(dV)$ whenever the support of f is assumed to be included in $\{z \in \mathbb{C}_+,\ \Re e(z) > 0\}$.

Remark 7. *This result is also true with P_α, i.e. when K is replaced by K_α and dV by dV_α.*

Proof. The proof of the boundedness of $|T_K|$ follows easily from Remark 4. One has only to prove the necessary condition on non negative f to get $|T_K|f$ in L^1.

Assume that f is such that $|T_K|f \in L^1(\mathbb{C}_+)$. We will use the same steps as in the proof of Lemma 2. We first write that the integral of $|T_K|f$ on the set $\{z = x+iy, y < 1\}$ is finite. We know that the term in $|z+i|^{-2}$ has a finite integral on this set. So we have

$$\int_{y<1}\int_{\mathbb{R}}\int_{\mathbb{C}_+} |z-\overline{\zeta}|^{-2} f(\zeta) d\zeta dz < \infty.$$

We are dealing with positive functions so that we can use Fubini's Theorem. This reduces to

$$\int_{\mathbb{C}_+} f(\zeta)\left(\int_0^1\int_{\mathbb{R}} \frac{dxdy}{(x-u)^2+(y+v)^2}\right) dV(\zeta) = \pi\int_{\mathbb{C}_+} f(\zeta)\ln\left(1+\frac{1}{v}\right) dV < \infty.$$

Hence

$$\int_{\mathbb{C}_+} f(\zeta)\ln\left(e+\frac{1}{\Im m(\zeta)}\right) dV(\zeta) < \infty.$$

Next, we consider the integral on $y > 1$ and prove that

$$\int_{\mathbb{C}_+} f(\zeta)\ln\left(e+|\zeta|\right) dV(\zeta) < \infty$$

under the additional assumption that the support of f is included in the set $\{z \in \mathbb{C}_+,\ \Re e(z) > 0\}$. We will use the fact that

$$\int_{x>0,y>1} |T_K|f(-x+iy)|\,dxdy < \infty.$$

Since the difference $(x^2+y^2)^{-1} - (x^2+(y+1)^2)^{-1}$ is integrable on this domain, it follows that

$$\left|\int_{u,v>0}\left[((x+u)^2+(y+v)^2)^{-1} - (x^2+y^2)^{-1}\right] f(u+iv)dudv\right|$$

is integrable on the quadrant $x, y > 0$. But the bracket is always negative, so that we are led to find some subset E_ζ for which

$$I_\zeta := \int_{E_\zeta}\left[|z|^{-2} - |z+\zeta|^{-2}\right] dxdy \geq c\ln(e+|\zeta|).$$

Take for E_ζ the set of z in the quadrant such that $1 < |z| \leq |\zeta|/2$, so that

$$I_\zeta \geq \frac{3}{4} \int_{E_\zeta} |z|^{-2} dV(z) \geq \frac{3\pi}{8} \ln(|\zeta|/2).$$

We conclude directly. □

4. Bloch functions, multipliers, duality

4.1. **Smoothness of L^∞-type.** We will show that Theorem 3 may be seen as a byproduct of the fact that the Bergman projector maps $L^\infty(\mathbb{C}_+)$ into the Bloch class. Indeed, let us recall the definition and some basic properties of the Bloch space. The Bloch space $\mathcal{B}(\mathbb{C}_+)$ is the space of holomorphic functions such that

$$\|f\|_{\mathcal{B}} := \sup_z \Im m(z)|f'(z)| < \infty.$$

It is a semi-norm on $\mathcal{B}(\mathbb{C}_+)$, but a norm on $\dot{\mathcal{B}}(\mathbb{C}_+) = \mathcal{B}/\mathbb{C}$, whose elements are equivalence classes modulo constants. Recall that $\dot{\mathcal{B}}(\mathbb{C}_+)$ identifies with the dual of $A^1(\mathbb{C}_+)$. The fact that constants are in the equivalence class of 0 is explained by the fact that functions in $A^1(\mathbb{C}_+)$ have mean 0. The pairing between $f \in A^1(\mathbb{C}_+)$ and $g \in \mathcal{B}(\dot{\mathbb{C}}_+)$ is given by

$$\langle f, g \rangle_{A^1, \mathcal{B}} = \int_{\mathbb{C}_+} f(z)\overline{g'(z)}(\Im m(z))\, dV(z). \tag{4.1}$$

We will also be interested by the space of Bloch functions itself and consider the norm on it given by

$$\|f\|_{\mathcal{B}}^+ := |f(i)| + \|f\|_{\mathcal{B}}.$$

The following lemma is elementary:

Lemma 3. *Bloch functions satisfy the inequality*

$$|f(z)| \leq C\|f\|_{\mathcal{B}}^+ \,\omega(z), \tag{4.2}$$

where ω is given by formula (1.1)

Proof. Indeed, let $z = x + iy$. For $|x| \leq y$, we just write

$$|f(z) - f(i)| \leq |f(x+iy) - f(iy)| + |f(iy) - f(i)| \leq C + C|\ln y|.$$

For $|x| \geq y$, we write

$$\begin{aligned}|f(z) - f(i)| \leq & |f(x+iy) - f(x+i|x|)| + |f(x+i|x|) - f(i|x|)| \\ & + |f(i|x|) - f(iy)| + |f(iy) - f(i)|.\end{aligned}$$

The two new terms are bounded by $C\ln(\frac{|x|}{y})$, and we conclude easily. □

As a direct consequence, we have the following corollary.

Corollary 1. *The pointwise product of a function in the Bergman space $A^1(\mathbb{C}_+)$ and a function in the Bloch space belongs to the space $A^1_{\omega^{-1}}(\mathbb{C}_+)$. Moreover, for $f \in A^1(\mathbb{C}_+)$ and $g \in \mathcal{B}(\mathbb{C}_+)$, one has*

$$\|fg\|_{A^1_{\omega^{-1}}} \lesssim \|f\|_{A^1} \|g\|_{\mathcal{B}}^+.$$

As described in the introduction, we will find later on a better result, namely those products are in $A_{\log}(\mathbb{C}_+)$, see Proposition 13.

Next, it is classical that the Bergman projector maps $L^\infty(\mathbb{C}_+)$ into the Bloch space, and so into the space of functions f that satisfy (4.2). To prove this, one has first to give a meaning to the Bergman projection of a bounded function. Classically, one considers the operator Q with kernel

$$(z,\zeta) \mapsto K(z,\zeta) - K(i,\zeta),$$

which is an integrable kernel in the variable ζ by Lemma 2. It is easy to verify that Q is the adjoint operator of T_K, which gives a proof by duality of Theorem 3.

4.2. **Multipliers.** Let us take the following definition.

Definition 1. *Let $\mathcal{B}_{\log}(\mathbb{C}_+)$ be defined by*

$$\mathcal{B}_{\log}(\mathbb{C}_+) := \{g \in \mathcal{H}(\mathbb{C}_+) : \exists\, C > 0 \text{ s.t. } \Im m(z)|g'(z)| \leq C\omega^{-1}(z)\ \forall z \in \mathbb{C}_+\}.$$

As for BMO functions, this space enters into the characterization of multipliers of the Bloch functions. A holomorphic function g is called a multiplier of the Bloch functions whenever, for any f in $\mathcal{B}(\mathbb{C}_+)$, the product fg also a Bloch function. A simple use of the closed graph theorem allows us to see that for such a multiplier, which is automatically a Bloch function, there exists a constant C such that the following inequality is satisfied.

$$\|fg\|_{\mathcal{B}}^+ \leq C\|f\|_{\mathcal{B}}^+. \tag{4.3}$$

The next proposition is reminiscent of the theorem on BMO multipliers of [11] in the unit disc and of [10] in the unbounded setting.

Proposition 3. *Multipliers of the Bloch space consist in bounded functions that are in the class $\mathcal{B}_{\log}$.*

Proof. It is clear that any function $g \in \mathcal{B}_{\log}$ gives a multiplier of the Bloch space: just write $(fg)' = f'g + g'f$. The first term satisfies the required inequality because g is bounded, the second one because of the condition

$$\Im m(z)|g'(z)| \leq C\omega^{-1}(z) \tag{4.4}$$

on g' and the estimate (4.2) on f.

Let us now prove that these conditions are necessary: let us take g satisfying (4.3) for any Bloch function f. First, let us prove that g is bounded. We claim that it is sufficient to prove that g is bounded in two cases: for $z = x + iy$, the first case is for those z such that $|z|y \geq 1$ and $|z| > 1$, so that $\omega(z) \simeq \ln(|z|)$; and the second one is for $|z|y \leq 1$ and $y < 1/2$, so that $\omega(z) \simeq \ln(1/y)$. Indeed, if $|z|y \geq 1$ but $|z| < 1$ or if $|z|y \leq 1$ but $y > 1/2$, then $\omega(z) \simeq 1$ and the boundedness of g is obtained by taking $f = 1$ and using (4.2).

So let us consider the two cases described above. In the first case, we take as test function f the function defined by $f(\zeta) = \log(\zeta)$ which has clearly a bounded Bloch norm. It allows us to get $|g(z)|\ln(|z|) \leq C\ln(|z|)$ hence g is bounded on this set. In the second case, we take $f(\zeta) = \log(\zeta - x) - \log(i - x)$, so that $f(i) = 0$. Using again (4.2), we find

$$|g(z)||\ln(y) - \ln(|i - x|) \leq C \ln(\frac{1}{y}).$$

But, because of the assumptions, $|\ln(y) - \ln(|i - x|)| = \ln(1/y) + \log(1 + 1/y) \leq C \ln(1/y)$, from which we conclude as before.

It remains to prove the bound on $|g'(z)|$ (4.4). As g is bounded and f is Bloch, we get from (4.3) that

$$\Im m(z)|g'(z)||f(z)| \leq \|fg\|_{\mathcal{B}} \leq C\|f\|_{\mathcal{B}}^{+}.$$

Then, we use the preceding test functions: for $|z| > 1$ and $|z|y > 1/2$, we take $f(z) = \log(z)$. As $\omega(z) \simeq \ln(|z|)$, on this set, we get $\Im m(z)|g'(z)| \lesssim \omega^{-1}(z)$. The second test function gives the expected estimate when $\omega(z) \simeq \ln(1/y)$. □

4.3. **Reproducing formula for $A^1_{\omega^{-1}}$-functions.** We first consider the weighted Bergman space $A^1_\rho(\mathbb{C}_+)$, with $\rho := (1 + \ln(e + 1/y))^{-1}$, i.e the space of all analytic functions f such that

$$\|f\|_{A^1_\rho} := \int_{\mathbb{C}_+} |f(x + iy)|\rho(y)\, dxdy < \infty.$$

It is clear that $A^1(\mathbb{C}_+)$ is continuously embedded in $A^1_\rho(\mathbb{C}_+)$.

The following result can be obtained in the same way as in [2, Proposition 1.3].

Proposition 4. (i) *There exists a constant $C > 0$ such that for all $x+iy \in \mathbb{C}_+$ and all $f \in A^1_\rho(\mathbb{C}_+)$, the following inequality holds.*

$$|f(x + iy)| \leq C \left(y^2\rho(y)\right)^{-1} \|f\|_{A^1_\rho}.$$

(ii) *There exists a constant $C > 0$ such that for all $y \in (0, \infty)$ and all $f \in A^1_\rho(\mathbb{C}_+)$,*

$$\|f_y\|_1 := \|f(\cdot + iy)\|_1 \leq C \left(y\rho(y)\right)^{-1} \|f\|_{A^1_\rho}.$$

We then obtain the following.

Proposition 5. *Let $f \in A^1_\rho(\mathbb{C}_+)$. Then*

(i) *The function $y \mapsto \|f_y\|_1$ is non-increasing and continuous on $(0, \infty)$.*
(ii) *$f(\cdot + i\varepsilon)$ is in $A^1_\rho(\mathbb{C}_+)$ for any $\varepsilon > 0$, and tends to f in $A^1_\rho(\mathbb{C}_+)$ as ε tends to zero.*

Proof. By assertion (ii) in Proposition 4, we have that f_y is in $H^1(\mathbb{C}_+)$, hence (i) holds.

Let us prove (ii): first using (i), we obtain

$$\|f(\cdot + i\varepsilon)\|_{A^1_\rho} = \int_0^\infty \|f_{y+\varepsilon}\|_1\rho(y)dy \leq \int_0^\infty \|f_y\|_1\rho(y)dy = \|f\|_{A^1_\rho}.$$

It is clear that f is the pointwise limit of $f(\cdot+i\varepsilon)$ as $\varepsilon\to 0$. Hence by (i) and [13, Theorem 5.6], we have that

$$\lim_{\varepsilon\to 0}\|f_{y+\varepsilon}-f_y\|_1=0$$

It follows from the above and the dominated convergence theorem that

$$\lim_{\varepsilon\to 0}\|f(\cdot+i\varepsilon)-f\|_{A^1_\rho}=\lim_{\varepsilon\to 0}\int_0^\infty\|f_{y+\varepsilon}-f_y\|_1\rho(y)dy=0.$$

□

Remark 8. *For any $\epsilon>0$, $z\mapsto f(z+i\epsilon)$ belongs to $H^1(\mathbb{C}_+)$ and, hence, is of integral* 0.

We now check the following.

Proposition 6. *$A^1(\mathbb{C}_+)$ is a dense subspace of $A^1_\rho(\mathbb{C}_+)$.*

Proof. Let $m>0$ be a large enough integer. Then for any $\varepsilon>0$, the function $g_\varepsilon(z)=(1-i\varepsilon z)^{-m}$ is in $A^1(\mathbb{C}_+)$.

Let $f\in A^1_\rho(\mathbb{C}_+)$. Define

$$F^{(\varepsilon)}(z)=g_\varepsilon(z)f(z+i\varepsilon).$$

Then by using assertion (ii) in Proposition 5, one sees that $F^{(\varepsilon)}$ belongs to $A^1_\rho(\mathbb{C}_+)$. Also observing that $y\mapsto(y^2\rho(y))^{-1}$ is non-increasing, one obtains from assertion (i) in Proposition 4 that the factor $f(z+i\varepsilon)$ is bounded and hence that $F^{(\varepsilon)}$ belongs to $A^1(\mathbb{C}_+)$.

Clearly, as $\varepsilon\to 0$, we have that $F^{(\varepsilon)}\to f$. That

$$\lim_{\varepsilon\to 0}\|F^{(\varepsilon)}-f\|_{A^1_\rho}=0$$

then also follows from the dominated convergence theorem. □

Proposition 7. *The set $A^1_\rho(\mathbb{C}_+)\cap A^1_{\omega^{-1}}(\mathbb{C}_+)$ is dense in $A^1_{\omega^{-1}}(\mathbb{C}_+)$.*

Proof. Let $f\in A^1_{\omega^{-1}}(\mathbb{C}_+)$. Consider $f_\varepsilon(z)=(1-i\varepsilon z)^{-1}f(z)$. Clearly, f_ε tends to f as ε tends to 0. Moreover, since $(1+\varepsilon|z|)^{-1}\rho(z)\le C_\varepsilon w(z)^{-1}$, we have that $f_\varepsilon\in A^1_\rho(\mathbb{C}_+)$. As $|f(z)-f_\varepsilon(z)|\le 2|f(z)|$, it follows from the dominated convergence theorem that $\|f-f_\varepsilon\|_{A^1_{\omega^{-1}}}\to 0$ as $\varepsilon\to 0$. □

We will need the following.

Proposition 8. *The Bergman projection P_α, $\alpha>0$, is bounded on $L^1_\rho(\mathbb{C}_+)$. Moreover, P_α reproduces the functions in $A^1_\rho(\mathbb{C}_+)$.*

Proof. That P_α maps $L^1_\rho(\mathbb{C}_+)$ boundedly into $A^1_\rho(\mathbb{C}_+)$ can be obtained as in Lemma 1. As P_α reproduces the elements of $A^1(\mathbb{C}_+)$, Proposition 6 allows us to conclude that this projector also reproduces functions in $A^1_\rho(\mathbb{C}_+)$. □

From Proposition 7 and Proposition 8, we obtain the following.

Corollary 2. *The Bergman projection P_α, $\alpha>0$, reproduces the functions in $A^1_{\omega^{-1}}(\mathbb{C}_+)$.*

4.4. **Duality.** We can now prove the following duality result.

Theorem 5. *The dual space* $\left(A^1_{\omega^{-1}}(\mathbb{C}_+)\right)^*$ *of* $A^1_{\omega^{-1}}(\mathbb{C}_+)$ *identifies with* $\mathcal{B}_{\log}$ *under the pairing*

$$\langle f, g\rangle_* = \int_{\mathbb{C}_+} f(z)\overline{g'(z)}(\Im m(z))\, dV(z), \tag{4.5}$$

$f \in A^1_{\omega^{-1}}(\mathbb{C}_+)$, *and* $g \in \mathcal{B}_{\log}(\mathbb{C}_+)$.

Proof. Let $g \in \mathcal{B}_{\log}(\mathbb{C}_+)$. Then that (4.5) defines a bounded linear operator is direct from the definition of the spaces involved.

Conversely, assume that Λ is an element of $\left(A^1_{\omega^{-1}}(\mathbb{C}_+)\right)^*$. Then Λ can be extended as an element $\tilde{\Lambda}$ of $\left(L^1_{\omega^{-1}}(\mathbb{C}_+)\right)^*$ with the same operator norm. Then classical arguments give that there is an element $h \in L^\infty(\mathbb{C}_+)$ such that for any $f \in L^1_{\omega^{-1}}(\mathbb{C}_+)$,

$$\tilde{\Lambda}(f) = \int_{\mathbb{C}_+} f(z)\overline{h(z)}\omega^{-1}(z)\, dV(z).$$

In particular, we have that for any $f \in A^1_{\omega^{-1}}(\mathbb{C}_+)$,

$$\Lambda(f) = \int_{\mathbb{C}_+} f(z)\overline{h(z)}\omega^{-1}(z)\, dV(z).$$

Using Lemma 1, we obtain that the projector P_2 is bounded from $L^1_{\omega^{-1}}(\mathbb{C}_+)$ into $A^1_{\omega^{-1}}(\mathbb{C}_+)$. It follows that for $f \in A^1_{\omega^{-1}}(\mathbb{C}_+)$,

$$\begin{aligned}\Lambda(f) &= \int_{\mathbb{C}_+} f(z)\overline{h(z)}\omega^{-1}(z)\, dV(z)\\ &= \int_{\mathbb{C}_+} P_2(f)(z)\overline{h(z)}\omega^{-1}(z)\, dV(z)\\ &= \int_{\mathbb{C}_+} f(z)\overline{P_2 g(z)}(\Im m(z))\, dV(z),\end{aligned}$$

where $g(z) = (\omega(z)\Im m(z))^{-1}h(z)$. We note that $g \in L^\infty_{\omega,1}(\mathbb{C}_+)$ and that using Proposition 2, one obtains that P_2 is bounded on $L^\infty_{\omega,1}(\mathbb{C}_+)$. It follows that if we define G to be a solution of $G'(z) = P_2 g(z)$, then $G \in \mathcal{B}_{\log}(\mathbb{C}_+)$. Hence

$$\Lambda(f) = \int_{\mathbb{C}_+} f(z)\overline{G'(z)}(\Im m(z))\, dV(z).$$

That is Λ is given by (4.5). The proof is complete. ☐

5. Products of functions in A^1 and $\mathcal{B}(\mathbb{C}_+)$ and Hankel operators

5.1. **Weak factorization.** Our aim is to prove the following result.

Theorem 6. *The product of a function in* $A^1(\mathbb{C}_+)$ *and a function in* $\mathcal{B}(\mathbb{C}_+)$ *is in* $A^1_{\omega^{-1}}(\mathbb{C}_+)$. *Conversely, a weak factorization result holds: if* f *is a holomorphic function in* $A^1_{\omega^{-1}}(\mathbb{C}_+)$, *then*

$$f = \sum g_j h_j,$$

with g_j's and h_j's holomorphic and such that

$$\|f\|_{L^1(\omega^{-1})} \simeq \sum_j \|g_j\|_{A^1} \|h_j\|_{\mathcal{B}}.$$

The first part of the result has already been given in Corollary 1. The bound above for $\|f\|_{L^1(\omega^{-1})}$ comes from the inequality given there. So we only have to concentrate on the weak factorization. The scheme of the proof is classical and starts from the atomic decomposition of a function $f \in A^1(\omega^{-1}dV)$, which we state now.

Proposition 9. *Let $f \in A^1_{\omega^{-1}}(\mathbb{C}_+)$. There exists a sequence of complex numbers $\{c_k\}$ and a sequence of points $\{w_k\}$ in $\mathbb{C}_+$ such that*

$$f(z) = \sum_{k=0}^{\infty} c_k \frac{(\Im m(w_k))^2 \omega(w_k)}{(z - \overline{w}_k)^4}$$

with $\sum |c_k| \lesssim \|f\|_{L^1(\omega^{-1})}$.

We leave its proof for the next section and concentrate on the weak factorization. Recall that functions

$$f(z) = \frac{(\Im m(w))^2 \omega(w)}{(z - \overline{w})^4} \tag{5.1}$$

are called atoms. We know that $\|f\|_{L^1(\omega^{-1})} \simeq 1$ as a consequence of Lemma 1 and Remark 3. As a consequence of the atomic decomposition, it is sufficient to factorize atoms, which we do now.

Proposition 10. *Let f be an atom given by* (5.1)*. There exist $g \in A^1(\mathbb{C}_+)$ and $\theta \in \mathcal{B}(\mathbb{C}_+)$ such that*

$$f = g \times \theta$$

with

$$\|g\|_{A^1} \times \|\theta\|^+_{\mathcal{B}} \lesssim 1.$$

Proof. The proof is in two steps. In the first one, we choose a non vanishing holomorphic function θ which satisfies $\|\theta\|^+_{\mathcal{B}} \lesssim 1$. In the second one, we prove that

$$\int_{\mathbb{C}_+} \frac{dV(z)}{|z-w|^4 |\theta(z)|} \lesssim \frac{\Im m(w)^{-2}}{\omega(w)}.$$

Constants must be independent of the choice of w. The second step is reminiscent of Lemma 1. We will use it and its proof.

First step: Choice of θ. Our choice will depend on $w = u + iv$.

(i) Assume first that $v < 1/2$ and $\ln(v^{-1}) \geq 3\ln(e+|w|)$ so that $\omega(w) \simeq \ln v^{-1}$. We choose

$$\theta : z \mapsto 1 - \log(z - \overline{w}) + \ln|w+i| + \log(i+z).$$

θ is a holomorphic function on $\mathbb{C}_+$ and the computation of its derivative and of its value at i leads to

$$\|\theta\|^+_{\mathcal{B}} := |\theta(i)| + \inf\{C > 0, (\Im m(z))|\theta'(z)| \leq C\} \leq 10.$$

Hence θ belongs to the Bloch space of the upper plane with a uniform norm. Moreover, when using the triangle inequality and the inequality $(a+b) \leq 2ab$, $a, b \geq 1$, we find that $\ln|z-\overline{w}| \leq \ln|w+i| + \ln|z+i| + \ln 2$, which leads to $\Re e(\theta(z)) > 1 - \ln 2$.

(ii) Assume that $\ln(v^{-1}) \leq 3\ln(e+|w|)$, so that $\omega(w) \simeq \ln(e+|w|)$. We choose

$$\theta(z) := 1 + \ln(i+z).$$

Again θ has a uniform Bloch norm. Furthermore, it satisfies

$$|\theta(z)| \geq \ln(e+|z|).$$

Second step: Estimate of g. In both cases,

$$g : z \mapsto \frac{(\Im m(w))^2 \omega(w)}{(z-\overline{w})^4 \theta(z)}$$

is a well defined holomorphic function in the upper half plane. We will prove that the inequality $\|g\|_{L^1} \lesssim 1$ is a consequence of lemma 1 with generalized to weights ω_1 and ω_2 (see Remark 2).

(i) Assume first that $v < 1/2$ and $\ln(v^{-1}) \geq 3\ln(e+|w|)$ so that $\omega(w) \simeq \ln(v^{-1}) \simeq \omega_1(w)$. We proceed as in the proof of Lemma 1 and recover the upper half plane as the union of E_k's, with $E_0 := Q_w$ and $E_k := Q_{w_k} \setminus Q_{w_{k-1}}, k \geq 1$. Recall that $w = u + iv$ and $w_k = u + i2^k v$.

On E_k, we use the the fact that $|z - \overline{w}| \leq 2\Im m(w_k)$, so that

$$\Re e(\theta(z)) \geq 1 - \ln 2 + \ln_+ \left(\frac{1}{|z-\overline{w}|}\right) \gtrsim \ln\left(e + (\Im m(w_k))^{-1}\right) = \omega_1(w_k).$$

Hence, proceeding as in the proof of Lemma 1, we have

$$\int |g(z)| dV(z) \lesssim \sum_k \frac{\omega_1(w)}{\omega_1(w_k)} 2^{-2k}.$$

We conclude by using the fact that $\omega_1(w_k) \geq \omega_1(w) - k\ln 2$.

(ii) It remains to consider the case when $\omega(w) \simeq \ln(e+|w|) = \omega_2(w)$. But then $|\theta(z)| \geq \ln(e+|z|) = \omega_2(z)$. So, the required estimate follows directly from Lemma 1 with ω_2 in place of ω. □

We now turn to the proof of the atomic decomposition.

5.2. **Proof of the atomic decomposition.** We prove the following atomic decomposition, which we state for atoms that may be other powers of the Bergman kernel.

Proposition 11. *Let f be a holomorphic function in $A^1_{\omega^{-1}}(\mathbb{C}_+)$. For any $\alpha > 0$, there exists a sequence of complex numbers $\{c_k\}$ and a sequence of points $\{w_k\}$ in $\mathbb{C}_+$ such that*

$$f(z) = c_\alpha \sum_{k=0}^{\infty} \frac{c_k (\Im m(w_k))^\alpha \omega(w_k)}{(z-\overline{w}_k)^{2+\alpha}}$$

with $\sum |c_k| \simeq \|f\|_{L^1(\omega^{-1})}$.

Proof. As a first remark, it follows from Lemma 1 that

$$\int_{\mathbb{C}_+} \frac{\Im m(w)^\alpha \omega(w)}{|z-\overline{w}|^{2+\alpha}\omega(z)} dV(z) \le C.$$

Hence, assuming $\sum |c_k|$ finite, we get that the function

$$z \mapsto \sum_{k=0}^{\infty} \frac{c_k(\Im m(w_k))^\alpha \omega(w_k)}{(z-\overline{w}_k)^{2+\alpha}}$$

belongs to $L^1(\omega^{-1}dV)$ with norm bounded by $\sum_{k=0}^{\infty} |c_k|$. It remains to prove that any function f in $A^1_{\omega^{-1}}(\mathbb{C}_+)$ may be written under this form.

The scheme of the proof is classical and goes back to the work by Coifman and Rochberg [7].

First, recall the notion of *η-lattice* in the terminology of [7]. The balls are in the Bergman metric (see [2] for instance).

Definition 2. *Suppose η is a given positive number less than 1 and $d(\cdot,\cdot)$ denote the Bergman distance in $\mathbb{C}_+$. We will call a sequence of points $\{\zeta_j\}$ in $\mathbb{C}_+$ an η-lattice if it satisfies the following properties:*

(1) *The balls $B_j := B(\zeta_j, \eta)$ cover C_+:*

$$\bigcup_j B_j = \mathbb{C}_+ \text{ (covering property).}$$

(2) *$\zeta_k \notin B_j$ whenever $j \neq k$.*
(3) *There exists $C > 0$ such that any $z \in \mathbb{C}_+$ does not belong to more than C different balls $B(\zeta_j, 2\eta)$ (overlapping property).*
(4) *The balls $\tilde{B}_j := B(\zeta_j, \eta/2)$ are disjoint:*

$$\tilde{B}_j \cap \tilde{B}_k = \emptyset \quad \text{whenever } j \neq k.$$

In other words, the balls centered at ζ_j and of radius η give a Whitney covering of $\mathbb{C}_+$.

The scheme of the proof of the atomic decomposition is to use an integral representation formula and to discretize it on an η-lattice. If η is sufficiently small then this produces a good approximation and iteration of the process yields the atomic decomposition. By Corollary 2, the operator P_α, $\alpha > 0$, reproduces $A^1_{\omega^{-1}}(\mathbb{C}_+)$-functions. Hence, for $f \in A^1_{\omega^{-1}}(\mathbb{C}_+)$

$$f(z) = \int_{\mathbb{C}_+} K_\alpha(z,w) f(w)\, dV(w)$$

where

$$K_\alpha(z,w) = c_\alpha \frac{\Im m(w)^\alpha}{(z-\overline{w})^{2+\alpha}}.$$

Now we use the η-lattice given before to find a covering of $\mathbb{C}_+$ by disjoint sets constructed by induction as follows:

$$E_0 := B_0 \setminus \left(\bigcup_{j=1}^{\infty} \tilde{B}_j \right).$$

$$E_j := B_j \setminus \left(\left(\bigcup_{k=0}^{j-1} E_k \right) \cup \left(\bigcup_{k=j+1}^{\infty} \tilde{B}_k \right) \right).$$

It is clear that $\tilde{B}_j \subset E_j \subset B_j$ and $\bigcup_{j=0}^{\infty} E_j = \mathbb{C}_+$, $E_j \cap E_k = \emptyset$ if $j \neq k$. Hence

$$f(z) = \sum_j \int_{E_j} K_\alpha(z,w) f(w) dV(w).$$

Let g be given by

$$g(z) := \sum_j K_\alpha(z,\zeta_j) f(\zeta_j)|E_j| = c_\alpha \sum_j \frac{(\Im m(\zeta_j))^\alpha}{(z-\overline{\zeta_j})^{2+\alpha}} f(\zeta_j)|E_j|.$$

Then,

$$\begin{aligned}
|f(z)-g(z)| &= \left| \sum_j \int_{E_j} \left(K_\alpha(z,w)f(w) - K_\alpha(z,\zeta_j)f(\zeta_j) \right) \, dV(w) \right| \\
&\leq \sum_j \int_{E_j} |K_\alpha(z,\zeta_j)| \omega(\zeta_j) |f(w)-f(\zeta_j)| \frac{dV(w)}{\omega(\zeta_j)} \\
&+ \sum_j \int_{E_j} |K_\alpha(z,w) - K_\alpha(z,\zeta_j)| |f(w)| dV(w) \\
&= I + II
\end{aligned}$$

One wants to prove that, for some universal constant $C>0$,

$$|f(z)-g(z)| \leq C\eta \sum_j \int_{B(\zeta_j,\eta)} |f(\zeta)| \omega^{-1}(\zeta) \, dV(\zeta) |K_\alpha(z,\zeta_j)| \omega(\zeta_j).$$

We first estimate the first term I. As usual we use the subharmonicity of the moduli of holomorphic functions to get, for $w \in E_j$,

$$|f(w)-f(\zeta_j)| \leq C\eta \frac{1}{|B(\zeta_j,\eta)|} \int_{B(\zeta_j,2\eta)} |f(z)| dV(z).$$

On the other hand, as $\Im m(z) \simeq \Im m(\zeta_j)$ and $|z| \leq 2|\zeta_j|$ for $z \in B(\zeta_j, 2\eta)$, one has $\omega(\zeta_j) \gtrsim \omega(z)$. Hence, it allows us to obtain

$$\frac{1}{\omega(\zeta_j)} |f(w)-f(\zeta_j)| \leq C\eta \frac{1}{|B(\zeta_j,\eta)|} \int_{B(\zeta_j,2\eta)} |f(z)| \frac{dV(z)}{\omega(z)},$$

hence

$$I \leq C\eta \sum_j \int_{B(\zeta_j,2\eta)} |f(\zeta)| \omega^{-1}(\zeta) \, dV(\zeta) |K_\alpha(z,\zeta_j)| \omega(\zeta_j).$$

For the second term II, we use that, for $w \in E_j$

$$|K_\alpha(z,w) - K_\alpha(z,\zeta_j)| \leq C\eta |K_\alpha(z,\zeta_j)|,$$

hence

$$\begin{aligned}\int_{E_j} |K_\alpha(z,w) - K_\alpha(z,\zeta_j)||f(w)|dV(w) &\leq C\eta|K_\alpha(z,\zeta_j)|\omega(\zeta_j)\int_{E_j}|f(w)|\frac{dV(w)}{\omega(\zeta_j)}\\ &\leq C\eta|K_\alpha(z,\zeta_j)|\omega(\zeta_j)\int_{E_j}|f(w)|\frac{dV(w)}{\omega(w)}.\end{aligned}$$

It allows us to get the expected inequality. Integrating it over $\mathbb{C}_+$ with respect to the measure $\omega^{-1}dV$, one gets by the overlapping property,

$$\|f-g\|_{L^1(\omega^{-1})} \leq C\eta\|f\|_{L^1(\omega^{-1})}. \tag{5.2}$$

Let A be the linear operator

$$A : \begin{cases} A^1_{\omega^{-1}}(\mathbb{C}_+) & \to & A^1_{\omega^{-1}}(\mathbb{C}_+) \\ f & \mapsto & g \end{cases}$$

then for η small enough, inequality (5.2) implies

$$\|I - A\|_{A^1_{\omega^{-1}} \to A^1_{\omega^{-1}}} \leq \frac{1}{2}$$

hence $(I-A)$ is a contraction so that A is invertible. Eventually, any $f \in A^1_{\omega^{-1}}(\mathbb{C}_+)$ may be written as

$$f = Ah = c_\alpha \sum_j \frac{(\Im m(\zeta_j))^\alpha}{(z-\overline{\zeta_j})^{2+\alpha}} h(\zeta_j)|E_j|$$

for some $h \in A^1_{\omega^{-1}}(\mathbb{C}_+)$ with

$$\|h\|_{A^1_{\omega^{-1}}} \lesssim \|f\|_{A^1_{\omega^{-1}}}.$$

It remains to prove that

$$\sum_j |h(\zeta_j)|\frac{|E_j|}{\omega(\zeta_j)} \lesssim \|f\|_{A^1_{\omega^{-1}}}.$$

We are going to prove that

$$\sum_j |h(\zeta_j)|\frac{|E_j|}{\omega(\zeta_j)} \lesssim \|h\|_{A^1_{\omega^{-1}}}.$$

From the subharmonicity of h,

$$|h(\zeta_j)|\frac{|E_j|}{\omega(\zeta_j)} \leq \frac{|E_j|}{|B_j|\omega(\zeta_j)}\int_{B_j}|h(w)|dV(w) \lesssim \int_{E_j}|h(w)|\frac{dV(w)}{\omega(w)}.$$

Summing on j gives the expected estimate. This completes the proof of Theorem 6. □

5.3. **Hankel operators.** Recall that the operator h_b is defined, for b in $L^\infty(\mathbb{C}_+)$ and f in $A^2(\mathbb{C}_+)$, by $h_b(f) := P(b\overline{f})$. To extend this definition, observe that, for any $g \in A^2(\mathbb{C}_+)$, one has also

$$\langle h_b(f), g\rangle = \langle b, fg\rangle.$$

This allows us to extend the definition of Hankel operators on $A^2(\mathbb{C}_+)$ to symbol in $\mathcal{B}(\mathbb{C}_+)$ by

$$\langle h_b(f), g\rangle = \langle b, fg\rangle_{A^1,\mathcal{B}}.$$

The notation $\langle\cdot,\cdot\rangle_{A^1,\mathcal{B}}$ stands for the duality bracket between $A^1(\mathbb{C}_+)$ and $\mathcal{B}(\mathbb{C}_+)$, and is given by (4.1).

It is well known that a necessary and sufficient condition on b to get a bounded operator on $A^2(\mathbb{C}_+)$ is b in the Bloch class. The next theorem gives a necessary and sufficient condition to extend h_b into a bounded operator on $A^1(\mathbb{C}_+)$.

Theorem 7. *Let b be in $\mathcal{B}(\mathbb{C}_+)$. Then the Hankel operator h_b extends to a bounded operator from $A^1(\mathbb{C}_+)$ to itself if and only if $b \in \mathcal{B}_{\log}(\mathbb{C}_+)$. Moreover,*

$$\|h_b\| \simeq \|b\|_{\mathcal{B}_{\log}}.$$

Proof. From the definition, if f and g in $A^2(\mathbb{C}_+)$,

$$\langle h_b(f), g\rangle = \langle b, fg\rangle_{A^1,\mathcal{B}}.$$

We claim that this formula makes sense also for $b \in \mathcal{B}_{\log}(\mathbb{C}_+)$, $f \in A^1(\mathbb{C}_+)$ and $g \in \mathcal{B}(\mathbb{C}_+)$. Indeed, if $b \in \mathcal{B}_{\log}(\mathbb{C}_+)$, $f \in A^1(\mathbb{C}_+)$ and $g \in \mathcal{B}(\mathbb{C}_+)$, the quantity

$$\int_{\mathbb{C}_+} f(z)g(z)\overline{b'(z)}\Im m z dV(z)$$

is well defined since $fg \in A^1_{\omega^{-1}}(\mathbb{C}_+)$. Moreover, the corresponding operator is bounded from $A^1(\mathbb{C}_+)$ into itself since

$$\left|\int_{\mathbb{C}_+} f(z)g(z)\overline{b'(z)}\Im m z dV(z)\right| \leq \|b\|_{\mathcal{B}_{\log}}\|f\|_{A^1}\|g\|_{\mathcal{B}}.$$

For the converse statement, we assume that h_b is well defined and bounded on $A^1(\mathbb{C}_+)$ and we prove that it is necessary for b to be in $\mathcal{B}_{\log}(\mathbb{C}_+)$. By Theorem 5, it is sufficient to prove that, for a dense subset, we have

$$|\langle b, F\rangle_{A^1_{\omega^{-1}},\mathcal{B}_{\log}}| = \left|\int_{\mathbb{C}_+} b'(z)\overline{F(z)}\Im m z dV(z)\right| \leq C\|F\|_{A^1_{\omega^{-1}}} \tag{5.3}$$

for some uniform constant. Consider the dense subset of functions in $A^1_{\omega^{-1}}(\mathbb{C}_+)$ with finite atomic decomposition,

$$F = \sum_{finite} a_j \frac{\Im m(\zeta_j)^2\omega(\zeta_j)}{(z-\zeta_j)^4},$$

with $\sum |a_j| \leq C\|F\|_{A^1_{\omega^{-1}}}$. We use the factorization of each term given in Proposition 10 to conclude that

$$\langle b, F\rangle_{A^1_{\omega^{-1}}, \mathcal{B}_{\log}} = \sum_{finite} a_j \langle b', f_j g_j\rangle_1$$

with

$$\langle b', f\rangle_1 := \int_{\mathbb{C}_+} b'(z)\overline{f(z)}\Im m(z) dV(z).$$

The boundedness of h_b on $A^1(\mathbb{C}_+)$ allows us to say that

$$|\langle b', f_j g_j\rangle_1| = |\langle h_b(f_j), g_j\rangle| \leq C$$

for some uniform constant C. Inequality (5.3) follows at once. □

Remark 9. *One may also define a Hankel operator on $\mathcal{B}(\mathbb{C}_+)$ as the adjoint operator of h_b acting on $A^1(\mathbb{C}_+)$. From the symmetry of the definition, this adjoint coincides with h_b and the analogous result holds.*

6. Associated Bergman-Musielak space

Here we are interested in the Bergman-Musielak space $A_{\log}(\mathbb{C}_+)$ and its links with $A^1_{\omega^{-1}}(\mathbb{C}_+)$.

Definition 3. *A holomorphic function f is in $A_{\log}(\mathbb{C}_+)$ if and only if*

$$\|f\|_{A_{\log}} := \inf\{\lambda > 0\ ;\ \int_{\mathbb{C}_+} \frac{|f(z)|/\lambda}{\ln(e + |f(z)|/\lambda) + \ln(e + |z|)} dV(z) \leq 1\}.$$

$\|\cdot\|_{A_{\log}}$ is a homogeneous quasi-norm. From the properties of the Musielak function

$$\Phi(t, z) := \frac{t}{\ln(e+t) + \ln(e + |z|)}$$

it is easy to deduce that f is in $A_{\log}(\mathbb{C}_+)$ if and only if

$$\int_{\mathbb{C}_+} \frac{|f(z)|}{\ln(e + |f(z)|) + \ln(e + |z|)} dV(z) < \infty.$$

Moreover, $\|f\|_{A_{\log}} \simeq 1$ if and only if

$$\int_{\mathbb{C}_+} \frac{|f(z)|}{\ln(e + |f(z)|) + \ln(e + |z|)} dV(z) \simeq 1.$$

6.1. **Embedding.** In this part, we establish the inclusion of $A_{\log}(\mathbb{C}_+)$ into the weighted space $A^1_{\omega^{-1}}(\mathbb{C}_+)$, as well as the fact that $A^1_{\omega^{-1}}(\mathbb{C}_+)$ is the smallest space having this property.

Proposition 12. *The space $A_{\log}(\mathbb{C}_+)$ is continuously embedded in the weighted space $A^1_{\omega^{-1}}(\mathbb{C}_+)$.*

Proof. By homogeneity it is sufficient to prove the existence of a constant C such that

$$\int_{\mathbb{C}_+} \frac{|f(z)|}{\ln(e+y^{-1})+\ln(e+|z|)}\, dV(z) \leq C$$

whenever f is a holomorphic function that satisfies the inequality

$$\int_{\mathbb{C}_+} \frac{|f(z)|}{\ln(e+|f(z)|)+\ln(e+|z|)}\, dV(z) \leq 1.$$

Here $z = x+iy$. We prove pointwise inequalities between the two functions to integrate. We can replace $\ln(e+|f(z)|)$ by $\ln(e+|f(z)|^2)$ in the denominator. We will prove the inequality

$$\ln(e+|f(z)|^2)+\ln(e+|z|) \leq \ln(e+y^{-1})+\ln(e+|z|), \tag{6.1}$$

which is sufficient to conclude.

A simple computation proves that $\ln(e+|f|^2)$ is a subharmonic function. This implies the subharmonicity of any convex function of $\ln(e+|f|^2)$. Since the function $t\mapsto \frac{e^{t/2}}{t+K}$ is convex on $(0,\infty)$ for $K>1$, we deduce that

$$\frac{|f(z)|}{\ln(e+|f(z)|^2)+\ln(e+|z|)} \lesssim y^{-2}\int_D \frac{|f(\zeta)|}{\ln(e+|f(\zeta)|^2)+\ln(e+|z|)}\, dV(\zeta),$$

where D is the disc of center z and radius $y/2$. But $\ln(e+|z|)\simeq \ln(e+|\zeta|)$ on D. So

$$\frac{|f(z)|}{\ln(e+|f(z)|^2)+\ln(e+|z|)} \lesssim y^{-2}.$$

Inequality (6.1) is obvious when $|f(z)|<4$. For $t>4$, the inequality $\dfrac{t}{\ln(e+t^2)} \lesssim y^{-2}$ implies that $\ln t \lesssim \ln(y^{-2})$, from which we conclude for Inequality (6.1). Finally, $\|f\|_{A_{\omega^{-1}}} \lesssim 1$ as required. □

Next, we want to prove that the two spaces $A^1_{\omega^{-1}}(\mathbb{C}_+)$ and $A_{\log}(\mathbb{C}_+)$ have the same atoms. Namely, we have the following lemma, which says that an atom f of $A^1_{\omega^{-1}}(\mathbb{C}_+)$, which is such that $\|f\|_{A^1_{\omega^{-1}}} \simeq 1$, is also such that $\|f\|_{A_{\log}} \simeq 1$. It is sufficient to give the upper bound because of the inclusion.

Lemma 4. *Let f be the atom given by*

$$f(z) = \frac{(\Im m(w))^\alpha \omega(w)}{(z-\overline{w})^{2+\alpha}}$$

Then, there exists a constant $C > 0$ independent on w such that

$$\int_{\mathbb{C}_+} \frac{|f(z)|}{\ln(e + |f(z)|) + \ln(e + |z|)} \, dV(z) \leq C.$$

Proof. The proof is, once again, based on Lemma 1 and Remark 2. Assuming that $\omega(w) \simeq \ln(e + |w|)$, we use the fact that

$$\int_{\mathbb{C}_+} \frac{(\Im m(w))^\alpha \ln(e + |w|)}{(z - \overline{w})^{2+\alpha} \ln(e + |z|)} \, dV(z) \lesssim 1.$$

Assuming that $\omega(w) \simeq \ln_+(v^{-1})$ and $v < 1/2$, then $\ln(e + |f(z)|) \gtrsim 1 + \ln_+(1/y) + \ln_+(|z - w|^{-2})$ as long as both terms are strictly positive. We write $\mathbb{C}_+ = \cup E_j$ as in Lemma 1 and find that $\ln(e + |f(z)|) \gtrsim \ln_+(v^{-1}) - k$ when z is in E_k, as long as $k \leq v^{-1}/2$. We conclude as before by cutting the sum into two parts. □

As a consequence, we get the following corollary.

Corollary 3. *The space $A^1_{\omega^{-1}}(\mathbb{C}_+)$ is the smallest Banach space containing the Bergman-Musielak space $A_{\log}(\mathbb{C}_+)$. The two spaces have same dual $\mathcal{B}_{\log}(\mathbb{C}_+)$.*

Proof. As $A_{\log}(\mathbb{C}_+)$ is continuously embedded in $A^1_{\omega^{-1}}(\mathbb{C}_+)$, it is sufficient to prove that the dual of $A_{\log}(\mathbb{C}_+)$ coincides with the one of $A^1_{\omega^{-1}}(\mathbb{C}_+)$. As $A^1(\mathbb{C}_+)$ is included in $A_{\log}(\mathbb{C}_+)$, an element of its dual is represented by a function $b \in \mathcal{B}(\mathbb{C}_+)$ and satisfies the inequality, for some uniform constant C,

$$|\langle b, f\rangle| \leq C\|f\|_{A_{\log}}.$$

We have to prove that b is in $\mathcal{B}_{\log}(\mathbb{C}_+)$. We take an atom

$$f : z \mapsto \frac{\omega(w)(\Im m(w))}{(z - \overline{w})^3}$$

in $A_{\log}$ and use the previous lemma. It follows that, for some uniform constant C, the following quantity is bounded

$$|\langle b, f\rangle| = \omega(w)(\Im m(w)) \left| \langle b, \frac{1}{(z - \overline{w})^3}\rangle \right| \leq C.$$

But here, we recognize $(\Im m(w))\omega(w)|b'(w)|$. It ends the proof. □

6.2. **John–Nirenberg inequality and products.** From the inequality in Lemma 2, we also obtain a kind of John–Nirenberg inequality:

Lemma 5. *There exist constants λ, C' such that Bloch functions satisfy the inequality*

$$\int_{\mathbb{C}_+} \frac{\exp(\lambda\|f\|_{\mathcal{B}}^{+\,-1}|f(z)|)}{(1 + |z|)^3} \, dV(z) \leq C'. \tag{6.2}$$

Proof. By (4.2), we have

$$\exp(\lambda\|f\|_{\mathcal{B}}^{+\,-1}|f(z)|) \leq e^{C\lambda} \left(\frac{e\Im m(z) + 1}{\Im m(z)}\right)^{C\lambda} (|z| + e)^{C\lambda}.$$

We find that the integral is finite when λ is such that $C\lambda < 1$. □

6.3. **Pointwise products.** One of our interests lies in in products of functions that are respectively in the Bergman space $A^1(\mathbb{C}_+)$ and in the Bloch space. When proceeding as in [5],we deduce from Lemma 5 that one has the following embedding:

Proposition 13. *The pointwise product of a function in the Bergman space $A^1(\mathbb{C}_+)$ and a function in the Bloch space belongs to the space $A_{\log}(\mathbb{C}_+)$.*

Proof. The result is a direct consequence of Lemma 2.1 of [5]. We recall it for an easier reading. Let $M \geq 1$. The following inequality holds for $s, t > 0$,

$$\frac{st}{M + \ln(e + st)} \leq \mathrm{e}^{(t-M)} + s. \tag{6.3}$$

Assume that $f \in A^1(\mathbb{C}_+)$ and $g \in \mathcal{B}(\mathbb{C}_+)$, and both have norm bounded by 1. We want to prove that fg satisfies the inequality

$$\int_{\mathbb{C}_+} \frac{|f(z)||g(z)|}{\ln(e + |f(z)||g(z)|) + \ln(e + |z|)}\, dV(z) \leq C.$$

We use (6.3) with $t = \lambda|g(z)|, s = |f(z)|/\lambda$ and $M = 3\ln(e + |z|)$. Here λ is the constant in (6.2). ☐

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