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ON FRACTIONAL OPERATORS WITH MORE THAN ONE SINGULARITY

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Abstract. Let $0 \leq \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be an integral operator given by a kernel of the form

$$K(x,y) = k_1(x - A_1 y)k_2(x - A_2 y) \dots k_m(x - A_m y),$$

where A_i are invertible matrices and each k_i satisfies a fractional size and a generalized fractional Hörmander condition that depends on α. In this survey, written in honour to Eleonor Harboure, we collect several results about boundedness in different spaces of the operator $T_{\alpha,m}$, obtained along the last 35 years by several members of the Analysis Group of FAMAF, UNC.

1. Introduction

Pola was a tireless and an expert collaborator of FAMAF's Analysis Group. Her ability and enthusiasm were manifested throughout four decades: teaching courses (in particular, a beautiful one on harmonic analysis and Hardy spaces), introducing integral operator theory in the National Seminar of Mathematics, advising works according to our diverse interests, directing theses or collaborating to direct them. Frank, disinterested, always devoted to the work and willing to help anyone in need.

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In this survey of inequalities for fractional type operators we will make a historical recount of the main known results regarding the following integral operator

$$T_{\alpha,m} f(x) = \int_{\mathbb{R}^n} k_1(x - A_1 y)k_2(x - A_2 y) \cdots k_m(x - A_m y) f(y)\, dy, \qquad (1.1)$$

where k_i, $i = 1, \dots, m$ satisfy certain size and general Hörmander conditions that depends on α, A_i are invertible matrices and $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$.

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In 1988, F. Ricci and P. Sjögren [23] obtained the $L^p(\mathbb{H}_1)$ boundedness, $1 < p \leq \infty$, for a family of maximal operators $\mathcal{M}_\alpha$ defined on the surface $\{z = \alpha xy\}$ on the three dimensional Heisenberg group $\mathbb{H}_1$. Some of these operators arise in the study of the boundary behavior of Poisson integrals on the symmetric space $SL(\mathbb{R}^3)/SO(3)$. To get the principal result, they studied the boundedness, on $L^2(\mathbb{R})$ of the integral operator,

$$T_{\alpha,2}f(x) = \int_{\mathbb{R}} |x-y|^{-\alpha_1} |x+y|^{-\alpha_2} f(y)dy,$$

where $0 \leq \alpha < n$, $\alpha_1 + \alpha_2 = n - \alpha$ and $0 < \alpha_1, \alpha_2 < n$. They considered $\alpha = 0$ and $n = 1$.

Motivated by this result, T. Godoy and M. Urciuolo, studied these kind of operators. During 1993, in [14], they considered the operator $T_{\alpha,2}$ for $n \geq 1$, $0 < \alpha < n$ and obtained for $1 < p < \infty$, the strong type (p,p) boundedness and for $p = 1$ the weak type $(1,1)$.

In 1994, T. Godoy, L. Saal and M. Uricuolo [13] and in 1996 T. Godoy and M. Urciuolo [15] obtained similar results for more general kernels.

In 1999, T. Godoy and M. Urciuolo [16] considered fractional integral operator given by the kernels $k_i(y) = \sum_{j \in Z} 2^{\frac{jn}{q_i}} \varphi_{i,j}\left(2^j y\right)$, $1 < q_i < \infty$, $\frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_m} = 1 - \frac{\alpha}{n}$, for $0 \leq \alpha < n$ and the matrices $A_i = a_i I$, with $a_i \neq 0$ and $a_i \neq a_j$. The functions $\varphi_{i,j}$ satisfies certain q_i-regularity condition. In this work they proved the boundedness of $T_{\alpha,m} : L^p \to L^q$ with $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

A classical problem in harmonic analysis is to study the boundedness of operators taking into account absolutely continuous measures with respect to the Lebesgue measure. In 1972 B. Mukenhoupt [20] introduced de $\mathcal{A}_p$ weights. These weights characterized the strong type (p,p), $1 < p < \infty$ and the weak type (p,p), $1 \leq p < \infty$, boundedness of M, the classical Hardy–Littlewood maximal operator.

In general terms let consider the fractional maximal operator

$$M_{\alpha,r}f(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q |f(y)|^r dy \right)^{1/r},$$

where $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $0 \leq \alpha < n$, $1 \leq r < \infty$ and the supremum is taken over all cube Q with side parallel to axes. We observe that $M = M_{0,1}$, is the classical Hardy–Littlewood maximal operator. For $0 < \alpha < n$, $M_\alpha = M_{\alpha,1}$ is the classical fractional maximal operator.

In 1974 B. Muckenhoupt and R. Wheeden [21] proved that if w is a weight (i.e., w is a non negative function and $w \in L^1_{loc}(\mathbb{R}^n)$) then M_α is bounded from $L^p(w^p)$ into $L^q(w^q)$, for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, if and only if

$$[w]_{\mathcal{A}_{p,q}} = \sup_Q \left[\left(\frac{1}{|Q|} \int_Q w^q \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w^{-p'} \right)^{\frac{1}{p'}} \right] < \infty. \tag{1.2}$$

We say that $w > 0$ is in the class $\mathcal{A}_{p,q}$ if equation (1.2) holds. We write $w \in \mathcal{A}_p$ if $w^p \in \mathcal{A}_{p,p}$ and define $\mathcal{A}_\infty = \bigcup_{p>1} \mathcal{A}_p$.

In 2005 [24], 2013 [25] and 2014 [26], M. S. Riveros and M. Urciuolo considered the operator given in equation (1.1) for

$$k_i(x) = \frac{\Omega_i(x')}{|x|^{\alpha_i}},$$

where $\Omega_i \in L^{p_i}(S^{n-1})$ satisfy appropriate p_i-Dini conditions and $\alpha_1 + \cdots + \alpha_m = n - \alpha$.

In this articles the authors gave an appropriate weighted $L^p \to L^q$ estimate, the weighted BMO and weak type estimates for certain weights in $\mathcal{A}_{p,q}$. These results were obtained by using the next Coifman type inequality:

$$\int_{\mathbb{R}^n} |T_{\alpha,m} f(x)|^p \, w^p(x)\, dx \leq C \sum_{i=1}^{m} \int_{\mathbb{R}^n} (M_{\alpha,s} f(x))^p \, w^p(A_i x)\, dx, \tag{1.3}$$

where $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{s} = 1$ and $w^p \in \mathcal{A}_\infty$.

Now, let $s < p < \frac{\alpha}{n}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w^s \in \mathcal{A}_{\frac{p}{s},\frac{q}{s}}$ and

$$w_{A_i}(x) := w(A_i x) \leq c w(x), \qquad \text{a.e. } x \in \mathbb{R}^n \text{ and } 1 \leq i \leq m, \tag{1.4}$$

for $f \in L^\infty_c$, from equation (1.3) and using the boundedness of the fractional maximal operator, we get

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |T_{\alpha,m} f(x)|^q \, w(x)^q \, dx \right)^{1/q} &\leq C \left(\int_{\mathbb{R}^n} (M_{\alpha,s} f(x))^q \, w(x)^q \, dx \right)^{1/q} \\ &\leq C \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p \right)^{1/p}. \end{aligned} \tag{1.5}$$

In Section 2 we will give some properties and results about these operators and some appropriated weights.

In 2011 and 2012 P. Rocha and M. Urciuolo in [28] and [27] obtained the boundedness of $T_{\alpha,m} : H^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $0 < p < 1/\alpha$ and $1/q = 1/p - \alpha$, where H^p is the classical Hardy space. They also showed that for these kind of operators, the $H^p - H^q$ boundedness can not be expect. This $H^p - H^q$ boundedness is true for the fractional integral operator.

Several results in variable $L^{p(\cdot)}$ spaces were obtained for these operators. P. Rocha and M. Urciuolo in [29] studied the $L^{p(\cdot)}$ into $L^{q(\cdot)}$ boundedness for $1/p(x) - 1/q(x) = \alpha/n$, where $p(x)$ is an appropriated radial function. Some years later, M. Urciuolo, L. A. Vallejos, and G. Ibañez Firnkorn, improved these results for more general kernels k_i and exponents $p(x)$ using extrapolation techniques. See [32], [31], [33] and [9].

This article continues in the following way. In Section 2 we give weighted boundedness results for the operator $T_{\alpha,m}$ and we present a suitable class of weights recently introduced in [12], which are good weights for the strong type (p, q) boundedness of the operator $T_{\alpha,m}$, for certain matrices A_i. In Section 3 we expose some recent results in variable Lebesgue spaces.

2. Weights and Weighted results for $T_{\alpha,m}$

At this point we turn our attention to the definitions of the fractional size and Hörmander conditions for the kernels that we will be working with along this paper. Let us establish the following notation. To introduce the operator that we consider in this paper, we first recall the following concepts.

A function $\Psi : [0,\infty) \to [0,\infty)$ is said to be a Young function if Ψ is continuous, convex, no decreasing and satisfies $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$.

For each Young function Ψ we can induce an average of the Luxemburg norm of a function f in the ball B defined by

$$\|f\|_{\Psi,B} := \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \Psi\left(\frac{|f|}{\lambda}\right) \leq 1\right\},$$

where $|B|$ is the Lebesgue measure of B. This function Ψ has an associated complementary Young function $\overline{\Psi}$ satisfying the generalized Hölder inequality

$$\frac{1}{|B|}\int_B |fg| \leq 2\|f\|_{\Psi,B}\|g\|_{\overline{\Psi},B}.$$

Also, for several Young functions we have the following Hölder inequality. If $\Psi_1, \dots, \Psi_m, \phi$ are Young functions satisfying $\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\phi^{-1}(t) \leq Ct$, for all $t \geq t_0$, some $t_0 > 0$ then

$$\|f_1 \cdots f_m g\|_{L^1,B} \leq C\|f_1\|_{\Psi_1,B}\cdots\|f_m\|_{\Psi_m,B}\|g\|_{\phi,B},$$

We say that the function ϕ is the complementary of the functions $\Psi_1, \dots, \Psi_m$.

We present now the fractional size condition and the generalized fractional Hörmander's condition. For more details, see [3] or [10].

Let Ψ be a Young function and let $0 \leq \alpha < n$. Let us introduce some notation: $|x| \sim s$ means $s < |x| \leq 2s$ and we write

$$\|f\|_{\Psi,|x|\sim s} = \|f\chi_{|x|\sim s}\|_{\Psi,B(0,2s)}.$$

The function K_α is said to satisfy the fractional size condition, $K_\alpha \in S_{\alpha,\Psi}$, if there exists a constant $C > 0$ such that

$$\|K_\alpha\|_{\Psi,|x|\sim s} \leq Cs^{\alpha-n}.$$

When $\Psi(t) = t$ we write $S_{\alpha,\Psi} = S_\alpha$. Observe that if $K_\alpha \in S_\alpha$, then there exists a constant $c > 0$ such that

$$\int_{|x|\sim s} |K_\alpha(x)|\,dx \leq Cs^\alpha.$$

The function K_α satisfies the $L^{\alpha,\Psi}$-Hörmander's condition $H_{\alpha,\Psi}$, if there exist constants $c_\Psi > 1$ and $C_\Psi > 0$ such that for all x and $R > c_\Psi|x|$,

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha}\|K_\alpha(\cdot - x) - K_\alpha(\cdot)\|_{\Psi,|y|\sim 2^m R} \leq C_\Psi.$$

We say that $K_\alpha \in H_{\alpha,\infty}$ if K_α satisfies the previous condition with $\|\cdot\|_{L^\infty,|x|\sim 2^m R}$ in place of $\|\cdot\|_{\Psi,|x|\sim 2^m R}$. When $\Psi(t) = t^r$, $1 \leq r < \infty$, we simply write $S_{\alpha,r}$ instead of $S_{\alpha,\Psi}$ and $H_{\alpha,r}$ instead of $H_{\alpha,\Psi}$. For $\alpha = 0$ we write $H_{0,r} = H_r$, the classical L^r-Hörmander condition.

Remark 2.1. *Observe that if $K_\alpha(x) = |x|^{n-\alpha}$ and A_1 is the identity matrix then $T_{\alpha,1} = I_\alpha$ the fractional integral and $K_\alpha \in S_{\alpha,\infty} \cap H_{\alpha,\infty}$.*

Remark 2.2. *Let $0 \leq \alpha < n$, $m \in \mathbb{N}$ and $1 \leq i \leq m$. Let $1 < r_i \leq \infty$, $s \geq 1$ defined by $\frac{1}{r_1} + \cdots + \frac{1}{r_m} + \frac{1}{s} = 1$ and $0 \leq \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. If $k_i \in S_{n-\alpha_i,r_i}$ for $1 \leq i \leq m$ then $K \in S_\alpha$ for $K(x,y) = k_1(x - A_1 y)k_2(x - A_2 y)\ldots k_m(x - A_m y)$. See details in* [10].

We will be considering matrices that satisfy the following hypothesis:

(H) A_i and $A_i - A_j$ are invertible for $1 \leq i, j \leq m$ and $i \neq j$.

The following theorem was proved in [26] by M. S. Riveros and M. Urciuolo and in a general version using the Young Hörmander conditions by G. Ibañez Firnkorn and M. S. Riveros in [10].

Theorem 2.3 ([26, 10])**.** *Let $0 \leq \alpha < n$ and $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by* (1.1)*. For $1 \leq i \leq m$, let $1 < r_i \leq \infty$, $s \geq 1$ defined by $\frac{1}{r_1} + \cdots + \frac{1}{r_m} + \frac{1}{s} = 1$ and $0 \leq \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. If $k_i \in H_{\alpha_i,r_i} \cap S_{n-\alpha_i,r_i}$ and suppose that the matrices A_i satisfy the hypothesis (H). If $\alpha = 0$, suppose $T_{0,m}$ be of strong type (p_0, p_0) for some $1 < p_0 < \infty$.*

Let $0 < p < \infty$; then there exists $C > 0$ such that, for $f \in L_c^\infty(\mathbb{R})$ and $w^p \in \mathcal{A}_\infty$,

$$\int_{\mathbb{R}^n} |T_{\alpha,m}f(x)|^p w^p(x)\, dx \leq C \sum_{i=1}^{m} \int_{\mathbb{R}^n} |M_{\alpha,s}f(x)|^p w^p(A_i x)\, dx,$$

whenever the left-hand side is finite. Furthermore, if w satisfies (1.4)*, then*

$$\int_{\mathbb{R}^n} |T_{\alpha,m}f(x)|^p w^p(x)\, dx \leq C \int_{\mathbb{R}^n} |M_{\alpha,s}f(x)|^p w^p(x)\, dx.$$

In this theorem what it is really proved is the following inequality

$$\int_{\mathbb{R}^n} |T_{\alpha,m}f(x)|^p w^p(x)\, dx \leq C \sum_{i=1}^{m} \int_{\mathbb{R}^n} |M_{\alpha,s}f(A_i^{-1}x)|^p w^p(x)\, dx.$$

If we consider $M_{\alpha,s,A^{-1}}$ the maximal operator defined by

$$M_{\alpha,s,A^{-1}}f(x) = M_{\alpha,s}f(A^{-1}x) = \sup_{Q \ni A^{-1}x} \left(\frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f|^s(y) dy \right)^{1/s}, \qquad (2.1)$$

then in Theorem 2.3 is obtained that the maximal operators that controls $T_{\alpha,m}$ are $M_{\alpha,s,A_i^{-1}}$, for $i = 1, \ldots, m$.

Therefore, it is natural to consider the following questions. Is there a characterization of the weights for the boundedness of $M_{\alpha,s,A^{-1}}$ defined in (2.1)? If this characterization is settled, are we able to obtain some weighted bounds as in (1.5)? For inequality (1.5) $w^p \in \mathcal{A}_\infty$ is required. Can such a condition be avoided?

To answer some of this questions in [12], G. Ibañez-Firnkorn, M. S. Riveros and R. Vidal considered the following classes of weights. We like to call them Marta's

(Urciuolo) weights, since she was the first to start trying to characterize this class of weights.

Let A be an invertible matrix and $1 \le p \le q < \infty$. A weight w is in the class $\mathcal{A}_{A,p,q}$, if

$$[w]_{\mathcal{A}_{A,p,q}} := \sup_Q \left(\frac{1}{|Q|} \int_Q w_A^q(x)\,dx \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w^{-p'}(x)\,dx \right)^{\frac{1}{p'}} < \infty, \qquad \text{for } p > 1,$$

$$[w]_{\mathcal{A}_{A,1,q}} := \sup_Q \left(\frac{1}{|Q|} \int_Q w_A^q(x)\,dx \right)^{\frac{1}{q}} \|w^{-1}\|_{\infty,Q} < \infty, \qquad \text{for } p = 1.$$

We write $w \in \mathcal{A}_{A,p}$ if $w^p \in \mathcal{A}_{A,p,p}$.

They give the following weak type (p,q) characterization:

Theorem 2.4 ([12]). *Let $0 \le \alpha < n$, $1 \le p < n/\alpha$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The maximal operator $M_{\alpha,\,A^{-1}}$ is bounded from $L^p(w^p)$ into $L^{q,\infty}(w^q)$ if and only if $w \in \mathcal{A}_{A,p,q}$. Even more $\|M_{\alpha,\,A^{-1}}\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \le C[w]_{\mathcal{A}_{A,p,q}}$.*

The proof follows similar steps as the one done by B. Muckenhoupt or B. Muckenhout and R. Wheeden. The condition $\mathcal{A}_{A,p,q}$ appears naturally.

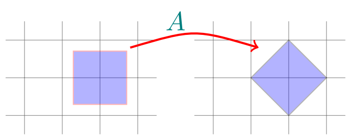
One of the difficulties in trying to repeat the properties of Muckenhoupt weights, is that we are not integrating the weight w in the same cube Q.

Observe that

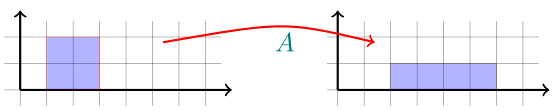
$$\left(\frac{1}{|Q|} \int_Q w_A^q(x)\,dx \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w^{-p'}(x)\,dx \right)^{\frac{1}{p'}} = \left(\frac{1}{|\det A|\,|Q|} \int_{AQ} w^q(y)dy \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w^{-p'}(x)\,dx \right)^{\frac{1}{p'}}.$$

We give some examples of matrices A, in order to show how they act in a cube Q.

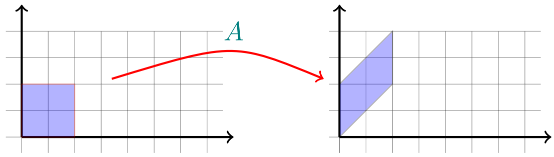
If A is the rotation in the angle $\frac{\pi}{4}$,



If A is a dilation in the x axis and a contraction in the y axis,



Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$,



Here we list some properties of the class of weights $\mathcal{A}_{A,p,q}$. Let $1 \leq p \leq q < \infty$, w be a weight and A be an invertible matrix:

(1) If $w \in \mathcal{A}_{p,q}$ and $w(Ax) \leq cw(x)$ a.e. $x \in \mathbb{R}^n$, then $w \in \mathcal{A}_{A,p,q}$.
(2) If $w \in \mathcal{A}_{A,p,q}$, then $w(Ax) \leq [w]_{\mathcal{A}_{A,p,q}} w(x)$ a.e. $x \in \mathbb{R}^n$.
(3) If $p < q$, $w \in \mathcal{A}_{A,p} \Rightarrow w \in \mathcal{A}_{A,q}$.
(4) If $w_0 \in \mathcal{A}_{A,1}$, $w_1 \in \mathcal{A}_{A^{-1},1}$, then $w = w_0 w_1^{1-p} \in \mathcal{A}_{A,p}$.
(5) The "A-doubling" property: Let $w \in \mathcal{A}_{A,p}$. For all $\lambda \geq 1$ and all Q we have
$$w_A(\lambda Q) \leq \lambda^{np} [w]_{\mathcal{A}_{A,p}} w(Q),$$
where λQ is the cube with same center as Q and size length λ times the side length of Q.
(6) If $w \in \mathcal{A}_{A,p} \cap \mathcal{A}_{A^{-1},p}$ then $w \in \mathcal{A}_p$ and $[w]_{\mathcal{A}_p} \leq [w]_{\mathcal{A}_{A,p}} [w]_{\mathcal{A}_{A^{-1},p}}$.

The items (1) and (6) show a relation between the class $\mathcal{A}_{A,p}$ and the Muckenhoupt class satisfying the condition $w(Ax) \leq cw(x)$ a.e. $x \in \mathbb{R}^n$.

The weight w satisfies the condition $w_A \lesssim w$ if $w(Ax) \leq cw(x)$ a.e. $x \in \mathbb{R}^n$.

An open question is if there exists a matrix A such that $\mathcal{A}_{A,p}$ is greater than
$$A_p \cap \{w_A \lesssim w\}.$$

From item (6) it follows that

(a) If $A^{-1} = A$ and $w \in \mathcal{A}_{A,p}$ then $w \in \mathcal{A}_p$ and $[w]_{\mathcal{A}_p} \lesssim [w]^2_{\mathcal{A}_{A,p}}$.
(b) If $A^N = A$ for some $N \in \mathbb{N}$ and $w \in \mathcal{A}_{A,p}$ then $w \in \mathcal{A}_p$ and $[w]_{\mathcal{A}_p} \lesssim [w]^N_{\mathcal{A}_{A,p}}$.

Some examples of matrices satisfying (a) and (b) are:

- A an ortonormal matrix, this is $A = A^t = A^{-1}$,
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$
- The rotation in $\theta = \frac{\pi}{4}$, $A = A^9$
$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Unfortunately they could not characterize the strong type (p, q) for these classes of weights for the maximal operator $M_{\alpha,A^{-1}}$. The reverse Hölder condition could not be obtained.

As an alternative way they started thinking the problem as a two-weight situation, for the pair (w_A, w). In [30] E. Sawyer introduced the following definition:

Definition 2.5. *Let $0 \leq \alpha < n$, $1 < p \leq q < \infty$, let (u,v) be a pair of weights. The pair $(u,v) \in \mathcal{M}_{\alpha,p,q}$ if it satisfies the testing condition*

$$[u,v]_{\mathcal{M}_{\alpha,p,q}} = \sup_Q \ v(Q)^{-1/p} \left(\int_Q M_\alpha(\chi_Q v)^q u \right)^{1/q} < \infty.$$

For the classical maximal operator M_α it is known the following two weight inequality, proved by E. Sawyer in [30] and the quantitative version by K. Moen in [19].

Theorem 2.6 ([30, 19])**.** *Let $0 \leq \alpha < n$, $1 < p \leq q < \infty$ and let (u,v) be a pair of weights. The following statements are equivalent:*

i. $(u,v) \in \mathcal{M}_{\alpha,p,q}$.
ii. *For every $f \in L^p(v)$,*

$$\left(\int_{\mathbb{R}^n} M_\alpha(fv)^q u \right)^{1/q} \leq C_{n,p,\alpha}[u,v]_{\mathcal{M}_{\alpha,p,q}} \left(\int_{\mathbb{R}^n} |f|^p v \right)^{1/p}.$$

Definition 2.7. *Let w be a weight, we say $w \in \mathcal{M}_{\alpha,A,p,q}$ if $(w_A^q, w^{-p'}) \in \mathcal{M}_{\alpha,p,q}$ and $w \in \mathcal{M}^*_{\alpha,A,q',p'}$ if $(w^{-p'}, w_A^q) \in \mathcal{M}_{\alpha,q',p'}$. For $\alpha = 0$ and $p = q$, we put $w \in \mathcal{M}_{A,p} := \mathcal{M}_{0,A,p,p}$.*

Remark 2.8. *Also if $w \in \mathcal{M}_{\alpha,A,p,q}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, we have that $w \in \mathcal{A}_{A,p,q}$ and $[w]_{\mathcal{A}_{A,p,q}} \leq [w]_{\mathcal{M}_{\alpha,A,p,q}}$.*

Then as a corollary of the previous theorem,

Corollary 2.9. [12] *Let $0 \leq \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The weight $w \in \mathcal{M}_{\alpha,A,p,q}$ if and only if*

$$\left(\int_{\mathbb{R}^n} M_{\alpha,A^{-1}}(g)^q w^q \right)^{1/q} \leq C_{n,p,\alpha}[w]_{\mathcal{M}_{\alpha,A,p,q}} \left(\int_{\mathbb{R}^n} |g|^p w^p \right)^{1/p},$$

for $g = fv = fw^{-p'} \in L^p(w^p)$, and where $[w]_{\mathcal{M}_{\alpha,A,p,q}} := [w_A^q, w^{-p'}]_{\mathcal{M}_{\alpha,p,q}}$.

Then there are the following inclusions:

$$\mathcal{A}_{p,q} \cap \{w : w_A \lesssim w\} \subset \mathcal{M}_{\alpha,A,p,q} \subset \mathcal{A}_{A,p,q},$$

and

$$\mathcal{A}_p \cap \{w : w_A \lesssim w\} \subset \mathcal{M}_{\alpha,A,p} \subset \mathcal{A}_{A,p}.$$

Now let us consider integral operator

$$T_{\alpha,2}f(x) = \int \frac{f(y)}{|x - A_1 y|^{\alpha_1} |x - A_2 y|^{\alpha_2}} dy.$$

The following strong type characterization with weights in the class $\mathcal{A}_{A,p,q}$ is obtained.

Theorem 2.10 ([12])**.** *Let $0 \leq \alpha < n$, $0 < \alpha_1, \alpha_2 < n$ such that $\alpha_1 + \alpha_2 = n - \alpha$. Let $1 < p < n/\alpha$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let A_1, A_2 be a invertible matrices such that $A_1 - A_2$ is invertible. Let $T_{\alpha,2}$ be the integral operator defined by* (1.1)*. Let w be a weight. If $T_{\alpha,2}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$ then $w \in \mathcal{A}_{A_1,p,q} \cap \mathcal{A}_{A_2,p,q}$. Furthermore, if $A_2 = A_1^{-1}$ or $A_1 = -I$ and $A_2 = I$, then $T_{\alpha,2}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$ if and only if $w \in \mathcal{A}_{A_1,p,q} \cap \mathcal{A}_{A_2,p,q}$.*

Remark 2.11. *In the proof of the first part of this theorem it is not used that $w \in \mathcal{A}_\infty$. In the second part when it is considered $A_2 = A_1^{-1}$ or $A_1 = -I$ and $A_2 = I$, $\mathcal{A}_\infty$ condition is obtained and it is used to prove the assumption.*

Remark 2.12. *The Theorem 2.10 contains some results given by E. Ferreyra and G. Flores in* [8]*, Theorem 3.2 and Corollary 3.3. They considered $p = q$, $1 < p < \infty$, $\alpha = 0$, $A_1 = -I$ and $A_2 = I$ and proved that if $w \in \mathcal{A}_p$ and $w(-x) \leq cw(x)$ a.e. $x \in \mathbb{R}^n$, then the operators $T_{0,2}$ is of strong type (p,p). Furthermore if $w^p(x) = |x|^\beta$, the operator $T_{0,2}$ is of strong type (p,p) if only if $w \in \mathcal{A}_p$.*

We now give the proof of Theorem 2.10, since it is self-contained. This result was obtained by G. Ibañez-Firnkorn in his doctoral thesis.

Proof of Theorem 2.10. Let $B = B(c_B, R)$ and $B_i = A_i^{-1}B$, $i = 1, 2$. Suppose that $f = \chi_{B_1}$ and $T_{\alpha,2}(\cdot v)$ is bounded from $L^p(v)$ into $L^q(u)$. Then

$$v(B_1)^{-1/p} \left(\int T_{\alpha,2}(\chi_{B_1} v)(x)^q u(x)\, dx \right)^{1/q} < \infty.$$

If $x \in B$ and $y \in B_1$, then

$$|x - A_1 y| \leq |x - c_B| + |c_B - A_1 y| \leq R + C_A R = (1 + C_A)R.$$

If $|x - A_2 y| \leq |x - A_1 y|$, then

$$|x - A_1 y|, |x - A_2 y| \lesssim R,$$

and since $\alpha_1 + \alpha_2 = n - \alpha$ then

$$\frac{v(y)}{|x - A_1 y|^{\alpha_1} |x - A_2 y|^{\alpha_2}} \geq \frac{v(y)}{|x - A_1 y|^{n-\alpha}} \geq C \frac{v(y)}{R^{n-\alpha}}.$$

If $|x - A_1 y| \leq |x - A_2 y|$, then

$$\frac{v(y)}{|x - A_1 y|^{\alpha_1} |x - A_2 y|^{\alpha_2}} \geq \frac{v(y)}{|x - A_2 y|^{n-\alpha}}.$$

If $2^j |x - A_1 y| \leq |x - A_2 y| \leq 2^{j+1} |x - A_1 y|$, then

$$\frac{1}{|x - A_2 y|^{n-\alpha}} \geq 2^{(\alpha-n)(j+1)} \frac{1}{|x - A_1 y|^{n-\alpha}} \geq 2^{(\alpha-n)(j+1)} \frac{1}{R^{n-\alpha}}.$$

In the case of $y \in B_1$ such that $|x - A_1 y| \leq |x - A_2 y|$, we have

$$\frac{v(y)\chi_{B_1 \cap \{|x-A_1y| \leq |x-A_2y|\}}}{|x - A_1 y|^{\alpha_1} |x - A_2 y|^{\alpha_2}} = \sum_{j=1}^{\infty} \frac{v(y)\chi_{\{2^j|x-A_1y| \leq |x-A_2y| \leq 2^{j+1}|x-A_1y|\}}(y)}{|x - A_1 y|^{\alpha_1} |x - A_2 y|^{\alpha_2}}$$
$$\geq C \frac{v(y)}{R^{n-\alpha}}.$$

Hence, if $x \in B$ and $y \in B_1$,

$$\frac{v(y)}{|x - A_1 y|^{\alpha_1} |x - A_2 y|^{\alpha_2}} \geq C_{n,\alpha,A} \frac{v(y)}{R^{n-\alpha}}.$$

We have an analogous result if $y \in B_2$.

If $x \in B$

$$T_{\alpha,2}(\chi_{B_1} v)(x) \geq R^{\alpha-n} v(B_1) = |B|^{\alpha/n-1} v(B_1).$$

Then we have

$$\begin{aligned}
v(B_1)^{-1/p} & \left(\int T_{\alpha,2}(\chi_{B_1} v)(x)^q u(x)\, dx \right)^{1/q} \\
& \geq v(B_1)^{-1/p} \left(\int_B T_{\alpha,2}(\chi_{B_1} v)(x)^q u(x)\, dx \right)^{1/q} \\
& \geq v(B_1)^{-1/p} \left(\int_B |B|^{q(\alpha/n-1)} v(B_1)^q u(x)\, dx \right)^{1/q} \\
& \geq v(B_1)^{-1/p} |B|^{\alpha/n-1} v(B_1) u(B)^{1/q} \\
& = |\det A_1^{-1}|^{-1/p'} |B|^{1/q} u(B)^{1/q} |B_1|^{1/p'} v(B_1)^{1/p'} \\
& = |\det A_1|^{1/p'} |B|^{\alpha/n-1} u(B)^{1/q} v_{A_1^{-1}}(B)^{1/p'}.
\end{aligned}$$

If we take $f = \chi_{B_2}$, in an analogous way we have

$$|B|^{\alpha/n-1} u(B)^{1/q} v_{A_2^{-1}}(B)^{1/p'} < \infty.$$

If $u = w^q$ and $v = w^{-p'}$ we conclude that if $T_{\alpha,2}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$ then $w \in \cap_{i=1}^2 \mathcal{A}_{A_i,p,q}$.

Furthermore, consider the case $A_1 = A$ and $A_2 = A^{-1}$. If $w \in \mathcal{A}_{A,p,q} \cap \mathcal{A}_{A^{-1},p,q}$ then $w_A \lesssim w$ and $w \in \mathcal{A}_{p,q}$. Therefore $T_{\alpha,2}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$. Now for the case $A_1 = -I$ and $A_2 = I$, if $w \in \mathcal{A}_{-I,p,q} \cap \mathcal{A}_{p,q}$ then $w_{-I} \lesssim w$ and $T_{\alpha,2}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$ (see [26]). □

In [12] the authors proved the $L^p(w^p)$ into $L^q(w^q)$ bounds for more general integral operator $T_{\alpha,m}$ defined in (1.1), with kernels satisfying a fractional size, $S_{\alpha,r}$, and a fractional Hörmander condition, $H_{\alpha,r}$. To obtain these results they used the sparse domination technique. In the last years this technique was used to obtain sharp weighted norm inequalities for singular or fractional integrals operators, for example see [18], [1].

In the case of the integral operator $T_{\alpha,m}$ with some particular matrices A_i they obtained a norm estimate relative to the constant of the weight.

Theorem 2.13 ([12]). *Let $0 \le \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by* (1.1). *For $1 \le i \le m$, let $\frac{n}{\alpha_i} < r_i \le \infty$ and $0 \le \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $k_i \in S_{n-\alpha_i,r_i} \cap H_{n-\alpha_i,r_i}$ and let the matrices A_i satisfy the hypothesis (H).*
If $\alpha = 0$, suppose that $T_{0,m}$ is of strong type (p_0, p_0) for some $1 < p_0 < \infty$.

Suppose that there exists $1 \le s < \frac{n}{\alpha}$ such that $\frac{1}{r_1} + \cdots + \frac{1}{r_m} + \frac{1}{s} = 1$, $A_j = A_i^{-1}$ for some $j \ne i$, and $w^s \in \bigcap_{i=1}^m \mathcal{A}_{A_i,\frac{p}{s},\frac{q}{s}}$, where $s < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then there exists $C > 0$ such that

$$\|T_{\alpha,m} f\|_{L^q(w^q)} \le C \|f\|_{L^p(w^p)} \sum_{i=1}^m [w^s]_{\mathcal{A}_{A_i,\frac{p}{s},\frac{q}{s}}}^{\max\left\{1-\frac{\alpha}{n}, \frac{(p/s)'}{q}\left(1-\frac{\alpha s}{n}\right)\right\}},$$

for all $f \in L^p(w^p)$.

Remark 2.14. *Observe that if $A_j = A_i^{-1}$ for some $j \ne i$ and $w^s \in \mathcal{A}_{A_i,\frac{p}{s},\frac{q}{s}} \cap \mathcal{A}_{A_j,\frac{p}{s},\frac{q}{s}}$, then $w^s \in \mathcal{A}_{\frac{p}{s},\frac{q}{s}}$ and $w_{A_j} \lesssim w$. Then, in this case, w^q and $w^{-s(p/s)'}$ belongs to $\mathcal{A}_\infty$.*

In the case $r_i = \infty$, for all $1 \le i \le m$, they obtained for the integral operator $T_{\alpha,m}$, a new two weighted result.

Theorem 2.15 ([12]). *Let $0 \le \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by* (1.1). *For $1 \le i \le m$, let $0 \le \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $k_i \in S_{n-\alpha_i,\infty} \cap H_{n-\alpha_i,\infty}$ and let the matrices A_i satisfy the hypothesis (H).*

If $\alpha = 0$, suppose that $T_{0,m}$ is of strong type (p_0, p_0) for some $1 < p_0 < \infty$. Let $1 < p \le q < \frac{n}{\alpha}$. If (u, v) are pairs of weights such that $(u_{A_i}, v) \in \mathcal{M}_{\alpha,p,q}$ and $(v, u_{A_i}) \in \mathcal{M}_{\alpha,q',p'}$ for $i = 1, \dots, m$, then the operator $T_{\alpha,m}$ is bounded from $L^p(v)$ into $L^q(u)$.

Given w a weight such that the pair $(u, v) = (w^q_{A_i}, w^{-p'})$ they arrived to the following corollary:

Corollary 2.16. [12] *Let $0 \le \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by* (1.1). *For $1 \le i \le m$, let $0 \le \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $k_i \in S_{n-\alpha_i,\infty} \cap H_{n-\alpha_i,\infty}$ and let the matrices A_i satisfy the hypothesis (H).*

If $\alpha = 0$, suppose that $T_{0,m}$ is of strong type (p_0, p_0) for some $1 < p_0 < \infty$. Let $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If w is a weight such that $w \in \mathcal{M}_{\alpha,A_i,p,q} \cap \mathcal{M}^*_{\alpha,A_i,q',p'}$, for $i = 1, \dots, m$, then the operator $T_{\alpha,m}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$.

Remark 2.17. *By Remark 2.8, if $w \in \bigcap_{i=1}^m \mathcal{M}_{\alpha,A_i,p,q}$ then $w \in \bigcap_{i=1}^m \mathcal{A}_{A_i,p,q}$ and this implies $w(A_i x) \le c w(x)$ a.e. $x \in \mathbb{R}^n$, for all $1 \le i \le m$.*

In [26] it was proved this same result of Corollary 2.16 under the hypothesis $w \in \mathcal{A}_{p,q}$ and $w(A_i x) \le c w(x)$ a.e. $x \in \mathbb{R}^n$ for all $1 \le i \le m$. The hypothesis $w \in \mathcal{A}_{p,q}$ and $w(Ax) \le c w(x)$ implies that $M_{\alpha,A^{-1}}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$. As it has been said in Corollary 2.9, if $w \in \mathcal{M}_{\alpha,A,p,q}$, then $w \in \mathcal{A}_{A,p,q}$.

Therefore the authors obtain a different proof without using essentially $w \in \mathcal{A}_\infty$.

3. Results in variable Lebesgue spaces for $T_{\alpha,m}$

In several works G. Ibañez Firnkorn, P. Rocha, M. Urciuolo and L. A. Vallejos proved different results about boundedness of the operator $T_{\alpha,m}$ define in (1.1) in variable Lebesgue spaces. See [29], [32], [31], [33] and [9]. We will highlight some general results.

Let $\mathcal{P}(\mathbb{R}^n)$ be the family of measure function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, let $L^{p(\cdot)}(\mathbb{R}^n)$ be the Banach space of measurable functions f on $\mathbb{R}^n$ such that, for some $\lambda > 0$, the modular

$$\rho_{p(\cdot)}(f/\lambda) = \int \left(\frac{f(x)}{\lambda}\right)^{p(x)} dx,$$

is finite, with norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : \ \rho_{p(\cdot)}(f/\lambda) \leq 1\}.$$

Let $p_- = \operatorname{ess\,inf} p(x)$ and $p_+ = \operatorname{ess\,sup} p(x)$.

These spaces are known as variable Lebesgue spaces and are a generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$. These spaces have been widely studied, see for example [4], [5] and [7]. The first step was to determine sufficient conditions on $p(\cdot)$ for the boundedness on $L^{p(\cdot)}$ of the Hardy–Littlewood maximal operator M. There exist several exponents for the boundedness of M, for example the classical log-Hölder conditions. We will focus in some of them. In [6] the authors define two conditions for the exponent functions.

Definition 3.1. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ we say that $p(\cdot) \in N_\infty(\mathbb{R}^n)$ if there exist constants $\Lambda_\infty > 0$ and $p^\infty \in [1, \infty]$ such that*

$$\int_\Omega \exp\left(-\Lambda_\infty \left|\frac{1}{p(x)} - \frac{1}{p^\infty}\right|^{-1}\right) dx < \infty,$$

where $\Omega = \left\{x \in \mathbb{R}^n : \left|\frac{1}{p(x)} - \frac{1}{p^\infty}\right| > 0\right\}$.

This condition appears for first time in [22] in the case of $p_+ < \infty$.

Remark 3.2. *Remark 4.6 in [6] affirms that $p(\cdot) \in N_\infty(\mathbb{R}^n) \Rightarrow p'(\cdot) \in N_\infty(\mathbb{R}^n)$.*

Definition 3.3. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ we say that $p(\cdot) \in K_0(\mathbb{R}^n)$ if there exists a constant C such that, for every cube Q,*

$$\|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{p'(\cdot)} \leq C\,|Q|\,.$$

This condition appears for first time in [2] in a more general setting.

Remark 3.4. *By the symmetry of the definition, $p(\cdot) \in K_0(\mathbb{R}^n) \Leftrightarrow p'(\cdot) \in K_0(\mathbb{R}^n)$.*

In [6] and [17] the authors proved that if $p(\cdot) \in N_\infty(\mathbb{R}^n) \cap K_0(\mathbb{R}^n)$, with $p_- > 1$, then M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

It is well known that certain conditions of continuity on $p(\cdot)$, such as the log-Hölder conditions $LH(\mathbb{R}^n)$, are sufficient for the boundedness of the maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$. But in [6] the authors showed that $LH(\mathbb{R}^n) \subset K_0(\mathbb{R}^n) \cap N_\infty(\mathbb{R}^n)$.

In 2022 G. Ibañez Firnkorn and L. A. Vallejos proved in [9] that the operator $T_{\alpha,m}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$, for $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$ and certain variable exponent $p(\cdot)$.

Theorem 3.5 ([9])**.** *Let $m \in \mathbb{N}$, $0 \leq \alpha < n$, $1 \leq r < \frac{n}{\alpha}$, $0 \leq \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$ and $\lambda \geq 0$. Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $r < p_- \leq p_+ < \frac{n}{\alpha}$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$. Let $T_{\alpha,m}$ be as in (1.1), with $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i}$ and let the matrices A_i satisfy the hypothesis (H). In the case of $\alpha = 0$, suppose that $T_{\alpha,m}$ is of strong type $(2,2)$. Suppose that $\varphi(t) = t^r \log(e+t)^\lambda$ is the complementary function of $\Psi_1, \ldots, \Psi_m$.*

(1) *If $p(A_i x) \leq p(x)$ a.e. $x \in \mathbb{R}^n$ and $\frac{q(\cdot)}{q_0} \in K_0(\mathbb{R}^n) \cap N_\infty(\mathbb{R}^n)$, where $q_0 = \frac{np_-}{n - \alpha p_-}$, then $T_{\alpha,m}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$.*
(2) *If $p(A_i x) = p(x)$ a.e. $x \in \mathbb{R}^n$ and the maximal operator is bounded on $L^{\left(\frac{q(\cdot)}{q_0}\right)'}(\mathbb{R}^n)$, where $q_0 = \frac{np_-}{n-\alpha p_-}$, then $T_{\alpha,m}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$.*

We notice that the results (1) and (2) are different. We know that the hypotheses $K_0(\mathbb{R}^n)$ and $N_\infty(\mathbb{R}^n)$ are sufficient conditions for the boundedness of maximal operator but are not necessary conditions.

Also, they obtain an end point estimate for $T_{0,m}$,

Theorem 3.6 ([9])**.** *Let $1 \leq r < \infty$, $\beta \geq 0$, $m \in \mathbb{N}$, $0 \leq \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n$. Let $T_{0,m}$ be a strong type $(2,2)$ operator defined as in (1.1), with $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i}$ and let the matrices A_i satisfy the hypothesis (H). Suppose that $\varphi(t) = t^r \log(e+t)^\beta$ is the complementary function of $\Psi_1, \ldots, \Psi_m$. Let f a bounded function with compact support. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 \leq r \leq p_- \leq p_+ < \infty$. If $\frac{p(\cdot)}{r} \in K_0(\mathbb{R}^n) \cap N_\infty(\mathbb{R}^n)$ and $p(A_i x) \leq p(x)$ a.e. $x \in \mathbb{R}^n$, then there exists $C > 0$ such that*

$$\left\| \chi_{\{x \in \mathbb{R}^n : |T_{0,m} f(x)| > \lambda\}} \right\|_{\frac{p(\cdot)}{r}} \leq C \left\| \frac{f}{\lambda} \right\|_{\frac{p(\cdot)}{r}}.$$

Remark 3.7. *The last theorem holds if we replace $p(A_i x) \leq p(x)$ and $\frac{p(\cdot)}{r} \in K_0(\mathbb{R}^n) \cap N_\infty(\mathbb{R}^n)$ by $p(A_i x) = p(x)$ and maximal operator bounded on $L^{\left(\frac{p(\cdot)}{r}\right)'}(\mathbb{R}^n)$.*

In the papers [32], [31] and [33] M. Urciuolo and L. A. Vallejos and in [29] M. Urciuolo and P. Rocha, obtained similar results in the case of smooth and rough kernels. The technique used in all these articles, is an adaptation of the classical Rubio de Francia's extrapolation Theorem, for variable exponent.

Finally, recall that given a locally integrable function b and a linear operator T, we define the commutator of T by

$$[b, T] f(x) = b(x) T(f)(x) - T(bf)(x).$$

The k-order commutator, $k \in \mathbb{N}$, by

$$T_b^k(f)(x) = [b, T_b^{k-1}] f(x).$$

In [11] G. Ibañez Firnkorn and M. S. Riveros proved strong and weak type inequalities for the commutator of $T_{\alpha,m}$, this is $T^k_{\alpha,m,b}$, in the classical weighted Lebesgue spaces.

In [9] the authors also proved for the commutator $T^k_{\alpha,m,b}$, the general case of Theorems 3.5 and 3.6, for variable Lebesgue spaces.

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