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EXTRAPOLATION OF COMPACTNESS FOR CERTAIN PSEUDODIFFERENTIAL OPERATORS

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Dedicated to the memory of Pola Harboure

Abstract. A recently developed extrapolation of compactness on weighted Lebesgue spaces is revisited and a new application to a class of compact pseudodifferential operators is presented.

1. Introduction

The extrapolation result of Rubio de Francia has become a powerful tool to extend the weighted boundedness of an operator from $L^{p_0}(w)$ for all $w \in A_{p_0}$ to all $L^p(w)$ with all $w \in A_p$. Extensions of the result to other contexts and situations have proven very useful too. For example, Eleonor Harboure (Pola) co-authored two collaborations, [11] and [3], which provided an extension to the off-diagonal case, allowing to deal with fractional integral like operators, and to operators and weights associated with the Schrödinger setting.

Very recently several authors have further taken on the task of extending the Rubio de Francia extrapolation result to the category of compact operators on weighted Lebesgue spaces. By works of Cao, Olivo and Yabuta [5], and Hytönen and Lappas [13, 12], linear, multilinear, and off-diagonal results have been established. The approaches by the two sets of authors are different. While both groups use a combination of interpolation and extrapolation results, the first one relies on the classical Frechet–Kolmogorov theorem about characterization of pre-compactness and the second one on more abstract arguments about compact operators.

Interpolation results of linear and multilinear compact operators have a long history going back at least to the seminal work of Calderón in [4]. On the other hand, the characterization of pre-compactness of sets in Lebesgue spaces received renewed attention in recent years. This was motivated in part by the applications to multilinear operators which started in [2], extending to the bilinear setting the

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result of Uchiyama [19] on the compactness of commutators of singular integrals with point-wise multiplication by functions in an appropriate subspace of BMO.

Essentially all recent applications in the literature about extrapolation of compact operators recover results about commutators which could be also obtained by other methods. We present here a new application related to the weighted compactness of certain pseudodifferential operators, extending an L^2 result of Cordes [7] in the non-weighted case. We take full advantage of the extrapolation result by proving the starting point in the extrapolation method on the Hilbert space $L^2(w)$, where matters are easily handled (some operators involved in the proof are actually Hilbert–Smith). We also point out that while preparing this manuscript, we became aware of the work of Stokdale, Villarroya, and Wick [17] using sparse domination and T1-compactness results for Calderón–Zygmund operators, which can be applied to the operators we consider too. Our methods, however, are rather straightforward and based on very simple computations and well-known results.

While little in this short article is then new, we hope our application puts in evidence one more time the strength of extrapolation results, which as mentioned earlier, was one of the many areas where Pola left her mark. We will fondly remember Pola, not only for her mathematical contributions but also for her warm, kind, and joyful personality. We dedicate this presentation to her memory.

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2. Preliminaries results

As usual, for $1 < r < \infty$, A_r will stand for the Muckenhoupt class of weights [14] defined by the condition

$$\|w\|_{A_r} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(r-1)}(x)\,dx\right)^{r-1} < \infty.$$

Let M be the Hardy–Litlewood maximal function

$$Mf(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

It is known that, for $1 < r < \infty$,

$$M : L^r(w) \longrightarrow L^r(w) \qquad \text{and} \qquad \|M\|_{L^r(w)} \lesssim \|w\|_{A_r}^{1/(r-1)}.$$

Another important property of A_r weights for our purposes is that they have at most polynomial growth at infinity. More precisely, we will need the following estimate. (Cf. also with estimates in the books by García Cuerva and Rubio de Francia [10, p. 412] or Stein [16, p. 209].)

Proposition 2.1. *For every $w \in A_2$, it holds that*

$$\left(\int_{\mathbb{R}^n}\frac{w(x)}{(1+|x|)^{2n}}\,dx\right)\left(\int_{\mathbb{R}^n}\frac{w^{-1}(x)}{(1+|x|)^{2n}}\,dx\right) \lesssim \|w\|_{A_2}^6.$$

Proof. For $x \in \mathbb{R}^n$, we have that $|y| < t$ implies $|y - x| < t + |x|$. Then,

$$M(\chi_{(-t,t)^n})(x) \geq \frac{1}{(t+|x|)^n} \int_{|y-x|<t+|x|} \chi_{(-t,t)^n}(y)\, dy \approx \left(\frac{t}{t+|x|}\right)^n .$$

Since M is bounded on $L^2(w)$ for $w \in A_2$, we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{w(x)}{(t+|x|)^{2n}}\, dx &\lesssim \frac{1}{t^{2n}} \|M(\chi_{(-t,t)^n}\|^2_{L^2(w)} \\
&\lesssim \frac{\|w\|^2_{A_2}}{t^{2n}} \|\chi_{(-t,t)^n}\|^2_{L^2(w)} \\
&\approx \|w\|^2_{A_2} \left(\frac{1}{t^n} \int_{|x|<t} w(x)\, dx\right)^2 .
\end{aligned}$$

Hence, since $w^{-1} \in A_2$, we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{2n}}\, dx\right) \left(\int_{\mathbb{R}^n} \frac{w^{-1}(x)}{(1+|x|)^{2n}}\, dx\right) \\
&\qquad \lesssim \|w\|^4_{A_2} \left(\frac{1}{t^n} \int_{|x|<t} w(x)\, dx\right)^2 \left(\frac{1}{t^n} \int_{|x|<t} w^{-1}(x)\, dx\right)^2 \lesssim \|w\|^6_{A_2},
\end{aligned}$$

and the result follows. □

Let us now define

$$M_\delta(f) = M(|f|^\delta)^{1/\delta}, \quad \delta > 0,$$

and

$$M^\# f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy, \quad M^\#_\delta f = M^\#(|f|^\delta)^{1/\delta}.$$

The maximal functions M_δ and $M^\#_\delta$ are related by a famous estimate due to Fefferman and Stein [9], which gives that, for all $0, < p, \delta < \infty$ and any $w \in \cup A_r$,

$$\int_{\mathbb{R}^n} M_\delta(f)(x)^p\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} M^\#_\delta(f)(x)^p\, w(x)\, dx. \tag{2.1}$$

We also recall that a bounded linear operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Calderón–Zygmund operator (CZO) if

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$$

for $x \notin \operatorname{supp} f$, where K satisfies

$$|\partial^\beta K(x,y)| \leq C_K |x-y|^{-n-|\beta|}, \quad |\beta| \leq 1. \tag{2.2}$$

Let T be a CZO. We define

$$\|T\|_{CZO} = \|T\|_{L^2 \to L^2} + C_K.$$

Álvarez and Pérez [1] showed that if T is a CZO operator, then, for any $0 < \delta < 1$,

$$M^\#_\delta(T(f))(x) \lesssim \|T\|_{CZO}\, M(f)(x). \tag{2.3}$$

(See also [15] for the $\|T\|_{CZO}$ dependance of the estimate.)

Remark 2.2. It is well known that if T is a CZO then T is also bounded in L^p for all $1 < p < \infty$ and also in $L^p(w)$ for any $w \in A_p$. Moreover if $\{T_j\}_j$ is a sequence of CZO so that for some CZO T, $\|T - T_j\|_{CZO} \to 0$ (so in particular $T_j \to T$ in L^2) then combining (2.1) and (2.3) we have that, for any $w \in A_p$,

$$\begin{aligned}\int_{\mathbb{R}^n} |(T - T_j)(f)(x)|^p w(x)\, dx &\leq \int_{\mathbb{R}^n} (M(|(T - T_j)(f)|^\delta)(x))^{p/\delta} w(x)\, dx \\ &\lesssim \int_{\mathbb{R}^n} (M_\delta^{\#}((T - T_j)(f))(x))^p w(x)\, dx \\ &\lesssim \|T - T_j\|_{CZO} \int_{\mathbb{R}^n} M(f)(x)^p w(x)\, dx,\end{aligned}$$

It follows that $\{T_j\}_j$ also converges to T in $L^p(w)$.

It is by now well known that the Rubio de Francia extrapolation theorem does not need an operator; it is about extrapolation of inequalities. More precisely we need the result in the following version.

Theorem 2.3. *Let $W > 0$ and let $\mathcal{F}$ be a family of pairs of measurable functions so that, for some $p_0 > 1$,*

$$\varphi(W) := \sup_{\|w\|_{A_{p_0}} \leq W} \sup_{(f,g) \in \mathcal{F}} \frac{\|g\|_{L^{p_0}(w)}}{\|f\|_{L^{p_0}(w)}} < \infty;$$

then, for every $p > 1$, there exist constants C_1 and C_2 so that if $M > 0$ satisfies that $C_2 M^{\max(1, \frac{p_0 - 1}{p-1})} = W$, then

$$\sup_{\|w\|_{A_p \leq M}} \sup_{(f,g) \in \mathcal{F}} \frac{\|g\|_{L^p(w)}}{\|f\|_{L^p(w)}} \leq C_1 \varphi(W) < \infty.$$

Proof. It is enough to follow, for example, the proof of Theorem 3.1 in [8]. □

We emphasize that the knowledge of the exact dependance of φ on the weight norm (or on p) is not needed for most applications and typically $g = T(f)$, where T is a linear operator.

The last ingredient for our approach is the Frechet–Kolmogorov theorem on the characterization of pre-compactness of subsets of Lebesgue spaces. Several versions have been obtained in recent years to treat the case of weighted Lebesgue spaces. For our purposes the following one by Clop and Cruz [6], which requires the weight to be in A_p, will suffice[1].

[1]More general versions of the theorem which do not require the weight to be in A_p exist and are particularly useful in the multilinear setting. See [20].

Theorem 2.4. *Let $1 < r < \infty$ and $w \in A_r$ and let $\mathcal{K} \subset L^r(w)$. If*

(i) *$\mathcal{K}$ is bounded in $L^r(w)$;*

(ii) $\lim\limits_{A\to\infty} \left(\int_{|x|>A} |f(x)|^r \, w(x)\, dx = 0 \right)^{1/r}$ *uniformly for $f \in \mathcal{K}$;*

(iii) $\lim\limits_{t\to 0} \|f(\cdot + t) - f\|_{L^r(w)} = 0$ *uniformly for $f \in \mathcal{K}$;*

then $\mathcal{K}$ is precompact in $L^r(w)$.

To use the Rubio de Francia extrapolation result we need to have some dependence on the boundedness of the operators on the weight norm which remains uniformly controlled to be able to combine it with the Frechet–Kolmogorov theorem. We find it convenient to introduce the following quantified version of compactness in the weighted situation.

Definition 2.5. *For $1 < r < \infty$, we say that a linear operator T is uniformly compact with respect to the class A_r if for some function $\varphi : (0,\infty) \to (0,\infty)$ the following hold.*

(i) For every $W > 0$,

$$\sup_{\|w\|_{A_r} \le W} \sup_{\|f\|_{L^r(w)} \le 1} \|Tf\|_{L^r(w)} \le \varphi(W).$$

(ii) For every $\varepsilon > 0$ and every $W > 0$, there exists $A > 0$ such that

$$\sup_{\|w\|_{A_r} \le W} \sup_{\|f\|_{L^r(w)} \le 1} \left(\int_{|x|\ge A} |Tf(x)|^r w(x)\, dx \right)^{1/r} < \varepsilon\varphi(W).$$

(iii) For every $\varepsilon > 0$ and every $W > 0$, there exists $\delta > 0$ so that, for every $|t| \le \delta$,

$$\sup_{\|w\|_{A_r} \le W} \sup_{\|f\|_{L^r(w)} \le 1} \left(\int_{\mathbb{R}^n} |Tf(x+t) - Tf(x)|^r w(x)\, dx \right)^{1/r} < \varepsilon\varphi(W).$$

Clearly, by Theorem 2.4, if T is uniformly compact with respect to the class A_r, then the image of the unit ball in $L^r(w)$ under T is pre-compact in $L^r(w)$ and hence T is compact. We can now apply Theorem 2.3 to show that such a T is actually uniformly compact in every $L^p(w)$ for $1 < p < \infty$ and $w \in A_p$. We mention again that results about extrapolation of compactness have already been obtained in [5] and [13]. We impose in our extrapolation result the stronger assumption of uniform compactness on the initial point p_0 and we obtain uniform compactness in the extrapolated values of p as well. An advantage to state the result in the following form is its extremely simple proof. As we will see, the uniform compactness assumption will be easily verified in the application we have in mind.

Theorem 2.6. *Suppose that T is uniformly compact with respect to the class A_{p_0} for some $1 < p_0 < \infty$. Then T is uniformly compact with respect to the class A_p for all $1 < p < \infty$.*

Proof. We need to prove *(i)–(iii)* in the definition of uniform compactness for all $1 < p < \infty$. We note that *(i)* is immediate from Theorem 2.3 by taking $g = T(f)$.

Next, to show *(ii)* for any p we observe that we have by hypothesis that given $\epsilon > 0$ and $W > 0$, there exist $A = A(\epsilon, W)$ such that, for all w with $\|w\|_{A_{p_0}} \le W$,

$$\frac{1}{\epsilon}\left(\int_{\mathbb{R}^n} |\chi_{\{|x|\ge A\}} T(f)|^{p_0} w\, dx\right)^{1/p_0} \le \varphi(W)\|f\|_{L^{p_0}(w)},$$

so we can let $g = \frac{1}{\epsilon}\chi_{\{|x|\ge A\}}T(f)$ and apply Theorem 2.3; that is, given $M > 0$, we can take $W = C_2 M^{\max(1, \frac{p_0-1}{p-1})}$ and hence we obtain that for all w such that $\|w\|_{A_p} \le M$,

$$\frac{1}{\epsilon}\left(\int_{\mathbb{R}^n} |\chi_{\{|x|\ge A\}} T(f)|^{p} w\, dx\right)^{1/p} \le C_1\phi(M)\|f\|_{L^{p}(w)},$$

where $\phi(M) = \varphi(C_2 M^{\max(1, \frac{p_0-1}{p-1})})$, which is enough for our purpose.

Similarly, to show *(iii)* we let $g = \epsilon^{-1}|(Tf)(.+t) - Tf(.)|$. □

3. Compact pseudodifferential operators

We will consider pseudodifferential operators of the form

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\xi}\, d\xi,$$

where the symbol σ will satisfy appropriate estimates. A classical result about compactness of pseudodifferential operators is given by the following theorem of Cordes [7].

Theorem 3.1. *Let T_σ be a pseudodifferential operator such that its symbol σ has continuous and bounded derivatives $\partial_x^\alpha \partial_\xi^\beta \sigma$ for all multi-indices $|\alpha|, |\beta| \le 2N$, where $N = [n/2] + 1$. Suppose also that*

$$(1-\Delta_x)^N (1-\Delta_\xi)^N \sigma(x,\xi) \to 0$$

as $|x|^2 + |\xi|^2 \to \infty$. Then T_σ is compact in L^2.

We note that while the conditions on the symbol give compactness in L^2 they are in general not strong enough to give even boundedness in L^p for $p \ne 2$ or more generally $L^p(w)$ for arbitrary $w \in A_p$. We are not aware of what could be minimal conditions to obtain compactness for $p \ne 2$, but to obtain continuity one typically needs to impose further decay of the derivatives of the symbol through, say, the membership of T_σ in some of the classical Hörmander classes $S^m_{\rho,\delta}$. Recall that a pseudodifferential operator $T_\sigma \in S^m_{\rho,\delta}$ if its symbol satisfies

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim_{\alpha,\beta} (1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}$$

for all multi-indices α, β. In particular, for the symbols of operators in $S^0_{1,0}$ we define the seminorms

$$p_{\alpha,\beta}(\sigma) = \sup_x \sup_\xi (1+|\xi|)^{|\beta|} |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)|.$$

The class $S^0_{1,0}$ is a subset of all bounded linear operators on L^p for all $1 < p < \infty$.

Using the extrapolation of compactness result we can prove the following.

Theorem 3.2. *Let T_σ be a pseudodifferential operator with symbol σ satisfying the conditions*

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim C_{\alpha,\beta}(x,\xi)(1+|\xi|)^{-|\beta|},$$

for all[2] *$|\alpha|, |\beta|$, where $C_{\alpha,\beta}$ is a bounded function which tends to zero as $|x|^2+|\xi|^2 \to \infty$.*

Then T_σ is compact in $L^p(w)$ for all $1 < p < \infty$ and all $w \in A_p$.

Proof. Note that T_σ obviously satisfies the hypothesis of Theorem 3.1. Following [7], we start by approximating σ by symbols of the form

$$\sigma_j(x,\xi) = \psi(2^{-j}x)\psi(2^{-j}\xi)\sigma(x,\xi),$$

where $\psi \in C_0^\infty$ nonnegative function identically equal to 1 in a neighborhood of the origin. It is shown in [7] that $\|T_{\sigma_j} - T_\sigma\|_{L^2 \to L^2} \to 0$ as $j \to \infty$. Clearly, $T_\sigma \in S^0_{1,0}$, and it is also easy to see that $T_{\sigma_j} \in S^0_{1,0}$ as well. It is well known that operators in such class are actually CZOs. Moreover the corresponding constants C_K in (2.2) are controlled by a finite sum of seminorms $p_{\alpha,\beta}$ of the corresponding symbols. See for example [18, Lemma 5.1.6].

We will show now that $p_{\alpha,\beta}(\sigma - \sigma_j) \to 0$ as $j \to \infty$. Fix α and β. We need to estimate terms of the form

$$S_{\alpha,\beta,\gamma,\lambda}(x,\xi) = |\partial_x^{\alpha-\gamma}\partial_\xi^{\beta-\lambda}(1-\psi(2^{-j}x)\psi(2^{-j}\xi))\partial_x^\gamma\partial_\xi^\lambda\sigma(x,\xi)|.$$

If all the derivatives fall on σ, then we easily obtain an estimate of the form

$$S_{\alpha,\beta,\gamma,\lambda}(x,\xi) \lesssim C_{\alpha,\beta}(x,\xi)(1+|\xi)^{-|\beta|}$$

for $(|x|^2+|\xi|^2)^{1/2} \gtrsim 2^j$ and $S_{\alpha,\beta,\gamma,\lambda}(x,\xi) = 0$ otherwise.

In the other cases, since at least one derivative falls on

$$(1-\psi(2^{-j}x)\psi(2^{-j}\xi)),$$

we can estimate the terms by

$$S_{\alpha,\beta,\gamma,\lambda}(x,\xi) \lesssim 2^{-(|\alpha-\gamma|+|\beta-\lambda|)j}C_{\gamma,\lambda}(x,\xi)(1+|\xi)^{-|\lambda|},$$

but noting also that $S_{\alpha,\beta,\gamma,\lambda}(x,\xi) = 0$ unless $|x| \approx 2^j$ if $|\alpha-\gamma| \neq 0$ and/or $|\xi| \approx 2^j$ if $|\beta-\lambda| \neq 0$. Hence, if only $|\alpha-\gamma| \neq 0$, we obtain

$$S_{\alpha,\beta,\gamma,\lambda}(x,\xi) \lesssim 2^{-|\alpha-\gamma|j}\,C_{\gamma,\beta}(x,\xi)(1+|\xi)^{-|\beta|},$$

[2]Obviously σ needs to satisfy the conditions in the theorem only for a sufficiently large but finite number of derivatives. We leave the computation of such number to the interested reader.

and certainly zero if $|x| \lesssim 2^j$, while if $|\beta-\lambda| \neq 0$ we have

$$\begin{aligned}
S_{\alpha,\beta,\gamma,\lambda}(x,\xi) &\lesssim 2^{-(|\alpha-\gamma|+|\beta-\lambda|)j}\, C_{\gamma,\lambda}(x,\xi)(1+|\xi|)^{-|\lambda|} \\
&\lesssim 2^{-|\beta-\lambda|j}\, C_{\gamma,\lambda}(x,\xi)(1+|\xi|)^{-|\lambda|} \\
&\lesssim |\xi|^{-|\beta-\lambda|}\, C_{\gamma,\lambda}(x,\xi)(1+|\xi|)^{-|\lambda|} \\
&\lesssim C_{\gamma,\lambda}(x,\xi)(1+|\xi|)^{-|\beta|}
\end{aligned}$$

if $|\xi| \approx 2^j$ and $S_{\alpha,\beta,\gamma,\lambda}(x,\xi) = 0$ otherwise. It follows that in all cases $p_{\alpha,\beta}(\sigma-\sigma_j) \to 0$ as $j \to \infty$. By Remark 2.2, it follows that $T_{\sigma_j} \to T_\sigma$ in $L^p(w)$ for all $1<p<\infty$ and $w \in A_p$.

To show that T_σ is compact in $L^p(w)$ it is enough then to show that each T_{σ_j} is. We will do so by showing that each T_{σ_j} is uniformly compact with respect to the class A_2 and then applying Theorem 2.6.[3]

Fix j. For simplicity in the notation, we write T_σ instead of T_{σ_j} but keeping in mind that the symbols now have compact support in some ball B in both x and ξ. We need to verify *(i)–(iii)* in Definition 2.5 for $r=2$. Note that *(i)* is just the boundedness of T_σ on $L^2(w)$, while *(ii)* is trivially satisfied because $T_\sigma f(x) = 0$ if x is outside a compact set.

Note that we can write

$$T_\sigma f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\sigma}^2(x, y-x) f(y)\, dy,$$

where $\widehat{\sigma}^2$ denotes the Fourier transform in the second variable. Then, for $|t|$ sufficiently small,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |T_\sigma f(x+t) - T_\sigma f(x)|^2 w(x)\, dx \\
&\quad = (2\pi)^{-n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (\widehat{\sigma}^2(x+t, y-x-t) - \widehat{\sigma}^2(x, y-x)) f(y) dy \right|^2 w(x)\, dx \\
&\quad \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\widehat{\sigma}^2(x+t, y-x-t) - \widehat{\sigma}^2(x, y-x))|^2 w(y)^{-1} dy\, w(x)\, dx\, \|f\|^2_{L^2(w)}.
\end{aligned}$$

We want to use the mean value theorem and so we need to estimate $|\nabla \widehat{\sigma}^2|$. We note first that the integration is already taking place in a compact set in x and that for $|y|$ sufficiently large $|y| \approx |y-x| \approx |y-x-t|$. Moreover by the estimates in the symbol, the function

$$(1+|z|)^{2n} |\nabla \widehat{\sigma}^2(u,z)|$$

[3]We note that the computations we shall perform easily show that each T_{σ_j} is actually a Hilbert–Smith operator in $L^2(w)$. This would suffice to show the compactness of T in $L^2(w)$ but to pass to other L^p with $p \neq 2$ we need the quantification of the uniform compactness.

is bounded, so we can continue the set of inequalities above with

$$\begin{aligned}&\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |t|^2 \frac{w(y)^{-1}}{(1+|y|)^{2n}} dy \, \frac{w(x)}{(1+|x|)^{2n}} \, dx \, \|f\|^2_{L^2(w)} \\ &\lesssim |t|^2 \varphi(\|w\|_{A_2})^2 \|f\|^2_{L^2(w)},\end{aligned}$$

by Proposition 2.1. This proves *(iii)* and concludes the proof of the theorem. □

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