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VERTICAL LITTLEWOOD–PALEY FUNCTIONS RELATED TO A SCHRÖDINGER OPERATOR

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Abstract. In this work we consider the Littlewood–Paley quadratic function associated to the Schrödinger operator $\mathcal{L} = -\Delta + V$ involving spatial derivatives of the semigroup's kernel. Under an appropriate reverse-Hölder condition on the potential we show boundedness on weighted L^p spaces for $1 < p < p_0$, where p_0 depends on the order of the reverse-Hölder property. Using a subordination formula we extend these results to the corresponding quadratic function associated to the semigroup related to $\mathcal{L}^\alpha$, $0 < \alpha < 1$.

1. Introduction

Our study is motivated by a recent manuscript of E. M. Ouhabaz (see [9]), regarding square functions related to the Schrödinger operator $\mathcal{L} = -\Delta + V$ on $\mathbb{R}^d$, with $d \geq 3$, where V is a non negative and locally integrable function. He points out that, in view of Trotter's formula the generated semigroup $e^{-t\mathcal{L}}$ is dominated by $e^{t\Delta}$ and, consequently, $\mathcal{L}$ possess a bounded holomorphic functional calculus which implies $L^p(\mathbb{R}^d)$ continuity of the so called horizontal Littlewood–Paley square functions, i.e.,

$$h_{\mathcal{L}} f(x) = \left(\int_0^\infty \left| t\mathcal{L} e^{-t\mathcal{L}} f(x) \right|^2 dt \right)^{1/2},$$

or the corresponding to the Poisson semigroup $\sqrt{\mathcal{L}}$,

$$g_{\mathcal{L}} f(x) = \left(\int_0^\infty \left| t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} f(x) \right|^2 dt \right)^{1/2}.$$

However, this behaviour is totally different when we treat with spatial derivatives instead of time derivative. More precisely, the vertical Littlewood–Paley functions are defined by

$$\mathcal{H}_{\mathcal{L}} f(x) = \left(\int_0^\infty \left| \nabla e^{-t\mathcal{L}} f(x) \right|^2 + \left| \sqrt{V} e^{-t\mathcal{L}} f(x) \right|^2 dt \right)^{1/2},$$

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or the coresponding to the Poisson semigroup $\sqrt{\mathcal{L}}$,

$$\mathcal{G}_{\mathcal{L}}f(x) = \left(\int_0^\infty t\left|\nabla e^{-t\sqrt{\mathcal{L}}}f(x)\right|^2 + t\left|\sqrt{V}e^{-t\sqrt{\mathcal{L}}}f(x)\right|^2 dt \right)^{1/2}.$$

For these vertical square functions he proves boundedness on L^p for $1 < p \leq 2$ assuming only $V \in L^1_{\text{loc}}$ and $V \geq 0$. Moreover, he shows that under those assumptions, boundedness for $p > 2$ fails for a large set of potentials.

In this work we show that there are also many potentials for which positive results hold, even for all p, $1 < p < \infty$, and we provide sufficient conditions on the potential V to get L^p continuity over an interval of the form $(1, p_0)$, $2 < p_0 \leq \infty$. Moreover, we obtain these results for a wide class of weights, including Muckenhoupt weights. Also, we obtain some properties for the endpoint case $p = 1$.

Our assumptions are those introduced by Shen (see [11]) and require the potential to satisfy a reverse Hölder inequality of order $q > d/2$, $d > 2$, where d stands for the dimension of the spatial variables. We remind that the last property means that there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B V^q \right)^{1/q} \leq \frac{C}{|B|}\int_B V, \tag{1}$$

holds for any ball $B \subset \mathbb{R}^d$. Given $q > 1$, the class of potentials satyisfing (1) will be denoted as RH_q. When the left hand side is replaced by $\sup_B V$, we say that $V \in RH_\infty$.

For such potentials, Shen introduced a key quantity known as the critical radius function $\rho : \mathbb{R}^d \longmapsto \mathbb{R}^+$ defined as

$$\rho(x) = \sup\left\{ r : \frac{r^2}{|B(x,r)|}\int_{B(x,r)} V \leq 1 \right\}.$$

An important feature of this function is the following pair of inequalities which control the variation of ρ when we go from a point to another, namely

$$c_\rho^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq c_\rho\, \rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}. \tag{2}$$

Appropriate families of weights have been introduced based on this function ρ. Following [3] we define, for a given $p > 1$, the class $A_p^\rho = \bigcup_{\theta \geq 0} A_p^{\rho,\theta}$, where $A_p^{\rho,\theta}$ is defined as those weights w such that

$$\left(\int_B w \right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}} \right)^{1/p'} \leq C|B|\left(1 + \frac{r}{\rho(x)}\right)^\theta$$

for every ball $B = B(x,r)$.

Similarly, when $p = 1$, we denote $A_1^\rho = \bigcup_{\theta \geq 0} A_1^{\rho,\theta}$, where $A_1^{\rho,\theta}$ is the class of weights w such that

$$\frac{1}{|B|}\int_B w \leq C\left(1 + \frac{r}{\rho(x)}\right)^\theta \inf_B w$$

for every ball $B = B(x,r)$. Finally let us say that we use the notation $A^\rho_\infty = \bigcup_{p\geq 1} A^\rho_p$.

We will deal first with the square function $\mathcal{H}_\mathcal{L}$. For convenience we shall consider separately

$$\mathcal{H}_S f(x) = \left(\int_0^\infty \left|\nabla e^{-t\mathcal{L}} f(x)\right|^2 dt\right)^{1/2}$$

and

$$\mathcal{H}_V f(x) = \left(\int_0^\infty \left|\sqrt{V} e^{-t\mathcal{L}} f(x)\right|^2 dt\right)^{1/2}.$$

In an analogous way we will break up $\mathcal{G}_\mathcal{L}$ into $\mathcal{G}_S$ and $\mathcal{G}_V$. Since the parts with spatial derivatives, $\mathcal{H}_S$ and $\mathcal{G}_S$, are much more involved than the others, our work is mainly concentrated on their study.

Now we are able to state one of our main results whose proof will be given in Section 3.

Theorem 1. *Let $V \in \mathrm{RH}_q$ for $q > d/2$ and $d \geq 3$. Then $\mathcal{H}_S$ is bounded on $L^p(w)$ for $1 < p < p_0$ with p_0 such that $\frac{1}{p_0} = \left(\frac{1}{q} - \frac{1}{d}\right)^+$ and w such that $w^{-\frac{1}{p-1}} \in A^\rho_{p'/p_0'}$. Moreover it is of weak type $(1,1)$ with respecto to w, i.e.,*

$$w(\mathcal{H}_S f(x) > \lambda) \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f| w,$$

for all $f \in L^1(w)$ and $\lambda > 0$; for weights satisfying $w^{p_0'} \in A^\rho_1$.

We point out that this result for $\mathcal{H}_S$ is quite different from that obtained for the Littlewood–Paley with time derivative $g_\mathcal{L}$, which is bounded for all L^p and all weigths A^ρ_p, regardless the size of q (see [3] and [1] for endpoint cases).

Regarding the operator $\mathcal{H}_V$ acting on $L^p(w)$ spaces, the result obtained for this operator is different than the one obtained for $\mathcal{H}_S$ since, if $V \in \mathrm{RH}_q$ with $q < \infty$ we can only obtain L^p boundedness for p in an interval. The precise result is stated in the following theorem whose proof is contained in Section 3.

Theorem 2. *Let $V \in \mathrm{RH}_q$ for $q > d/2$ and $d \geq 3$. Then $\mathcal{H}_V$ is bounded on $L^p(w)$ for $1 < p < 2q$ with w such that $w^{-1/(p-1)} \in A^{\rho,\infty}_{p'/(2q)'}$ and it is also bounded in $L^1(w)$ for w such that $w^{(2q)'} \in A^{\rho,\infty}_1$.*

Finally we turn our attention to the Littlewood–Paley related to the Poisson semi-group. In fact we consider a family of square functions associated to the semi-group of $\mathcal{L}^\alpha$, for $0 < \alpha \leq 1$, so that $\mathcal{G}_S$ is just the case $\alpha = 1/2$ and $\mathcal{H}_S$ the case $\alpha = 1$. Through a subordination formula we are able to extend all the results obtained for $\mathcal{H}_S$ to the whole family of square functions. The same techniques can be applied when considering the square functions involving the potential V. The precise definition and details of the proofs are contained in Section 4.

2. Preliminaries

In this section we develop some estimates that will be needed in the proofs of our main results. This results deal mostly with the heat kernel associated with $\mathcal{L}$, its derivatives and the relation between this kernel and the heat kernel associated with the Laplacian.

We begin by giving two auxiliary lemmas concerning the potential V. The first is an useful inequality for $V \in RH_q$ with $q > d/2$ that follows easily from Lemma 1.2 and Lemma 1.8 in [11].

In what follows, we will use that the reverse-Hölder condition implies the doubling property for V. This is, there exists $C > 0$ such that

$$\int_{2B} V \leq C \int_B V, \quad \text{for all ball} \quad B \subset \mathbb{R}^d.$$

The infimum of the constant satisfying the above çondition will be referred as the doubling constant of V.

Lemma 1. *Let $N_2 = \log_2 C_1 + 2 - d$, where C_1 is the doubling constant of V. Then, for any $x_0 \in \mathbb{R}^d$, $R > 0$,*

$$\frac{1}{R^{d-2}} \int_{B(x_0,R)} V(y)\, dy \leq C \left(1 + \frac{R}{\rho(x_0)}\right)^{N_2} \left(1 + \frac{\rho(x_0)}{R}\right)^{d/q-2}.$$

As a particular case of Corollary 2.8 in [6] we have the following lemma.

Lemma 2. *Let $\delta = 2 - d/q$ and $c > 0$. Then there exists $C > 0$, such that*

$$\frac{1}{t^{d/2}} \int_{\mathbb{R}^d} e^{-c\frac{|x-z|^2}{t}} V(z)\, dz \leq \frac{C}{t} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta}, \qquad \text{for } 0 < \sqrt{t} < \rho(x) \text{ and } x \in \mathbb{R}^d.$$

The following covering lemma can be found in [5]

Lemma 3. *There exists a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ such that the family of critical cubes given by $Q_j = Q(x_j, \rho(x_j))$ satisfies*

(a) $\bigcup_{j\in\mathbb{N}} Q_j = \mathbb{R}^d$.

(b) *There exist positive constants C and N_1 such that for any $\sigma \geq 1$, $\sum_{j\in\mathbb{N}} \mathcal{X}_{\sigma Q_j} \leq C\sigma^{N_1}$.*

In what follows we consider

$$e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^d} k_t(x,y) f(y)\, dy,$$

and

$$e^{t\Delta} f(x) = \int_{\mathbb{R}^d} h_t(x-y) f(y)\, dy.$$

Since V is non-negative, the Feynman–Kac formula implies

$$0 \leq k_t(x,y) \leq h_t(x-y) = \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{d/2}}. \tag{3}$$

In fact, under our assumptions on V, it is shown in [11] that for each $N > 0$ there exists a constant C_N such that

$$k_t(x,y) \leq \frac{C_N}{t^{d/2}} e^{-\frac{|x-y|^2}{5t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \tag{4}$$

for x, $y \in \mathbb{R}^d$ and $t > 0$. Another useful estimate (see [4]) assures that there exists $c > 0$ and for every $N > 0$ there exists a constant C_N such that

$$|\partial_t k_t(x,y)| \leq C_N \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{(d+2)/2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \tag{5}$$

for x, $y \in \mathbb{R}^d$ and $t > 0$.

In what follows we will denote

$$G(x,y) = \int_{B(x,|x-y|)} \frac{V(z)}{|x-z|^{d-1}}\, dz, \tag{6}$$

since this expression arise naturally throughout this work. As was shown in [11], if $V \in RH_q$ with $q > d$ then, by Hölder's inequality,

$$G(x,y) \leq \frac{C}{|x-y|^{d-1}} \int_{B(x,|x-y|)} V(z)\, dz.$$

This equation and Lemma 1 lead us to

$$G(x,y) \leq \frac{C}{|x-y|} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{N_2} \left(1 + \frac{\rho(x)}{|x-y|}\right)^{d/q-2}. \tag{7}$$

In the next lemma we give an estimate involving spatial derivative of k_t that improves Lemma 8 in [8]. We will use the notation $\nabla_1 F(x,y)$ when we take derivatives of $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, with respect to the first variable and $\nabla_2 F(x,y)$ with respect to the second.

Lemma 4. *Let $V \in \mathrm{RH}_q$ for some $q > d/2$. Then there exists $c > 0$ and for each $N > 0$, $C_N > 0$ such that*

$$|\nabla_1 k_t(x,y)| \leq C_{N,c} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \left(\frac{1}{|x-y|} + G(x,y)\right),$$

for all x, $y \in \mathbb{R}^d$ and $t > 0$.

Proof. Let Γ_0 be the fundamental solution of $-\Delta$, $t > 0$, x_0, $y_0 \in \mathbb{R}^d$ and u a solution of the equation

$$\partial_t u + \mathcal{L} u = 0. \tag{8}$$

Let $R = |x_0 - y_0|/8$ and $\eta \in \mathcal{C}_0^\infty(B(x_0, 2R))$ such that $\eta \equiv 1$ in $B(x_0, 3R/2)$, $|\nabla \eta| \leq C/R$ and $|\nabla^2 \eta| \leq C/R^2$. By (8)

$$\begin{aligned} -\Delta(u\eta) &= -\Delta u \cdot \eta - 2\nabla u \cdot \nabla \eta - u \cdot \Delta \eta \\ &= -(\partial_t u)\eta - V u \eta - 2\nabla u \cdot \nabla \eta - u \cdot \Delta \eta. \end{aligned}$$

Integrating by parts,

$$-\int \Gamma_0(x,z)\nabla_1 u(z,t)\cdot\nabla\eta(z)\,dz = \int \nabla_2\Gamma_0(x,z)\cdot\nabla\eta(z)u(z,t)\,dz + \int \Gamma_0(x,z)\cdot\Delta\eta(z)u(z,t)\,dz.$$

Therefore,

$$u(x,t)\eta(x) = \int \Gamma_0(x,z)\left[-V(z)u(z,t)\eta(z) - \eta(z)\partial_t u(z,t) + \Delta\eta(z)u(z,t)\right]dz + 2\int \nabla_2\Gamma_0(x,z)\cdot\nabla\eta(z)u(z,t)\,dz.$$

Now, for $x \in B(x_0,R)$,

$$\begin{aligned}|\nabla_1 u(x,t)| &= |\nabla_1 u(x,t)\eta(x)| \\ &\le \int_{B(x_0,2R)} \frac{V(z)|u(z,t)|}{|x-z|^{d-1}}\,dz + \int_{B(x_0,2R)} \frac{|\partial_t u(z,t)|}{|x-z|^{d-1}}\,dz \\ &\qquad + \frac{C}{R^{d+1}}\int_{B(x_0,2R)} |u(z,t)|\,dz \\ &\le \sup_{B(x_0,2R)}|u(\cdot,t)|\int_{B(x_0,2R)}\frac{V(z)}{|x-z|^{d-1}}\,dz + CR\sup_{B(x_0,2R)}|\partial_t u(\cdot,t)| \\ &\qquad + \frac{C}{R}\sup_{B(x_0,2R)}|u(\cdot,t)|.\end{aligned}$$

Let us take now $u(x,t) = k_t(x,y_0)$. Hence,

$$\begin{aligned}|\nabla_1 k_t(x_0,y_0)| \le \sup_{B(x_0,2R)}|k_t(\cdot,y_0)|\int_{B(x_0,2R)}\frac{V(z)}{|x_0-z|^{d-1}}\,dz \\ + CR\sup_{B(x_0,2R)}|\partial_t k_t(\cdot,y_0)| \\ + \frac{C}{R}\sup_{B(x_0,2R)}|k_t(\cdot,y_0)| \\ = I + II + III.\end{aligned}$$

Now, using (4), for each $N>0$ there is a constant C_N such that

$$I \le \frac{C_N}{t^{d/2}}e^{-\frac{|x_0-y_0|^2}{t}}\left(1+\frac{\sqrt{t}}{\rho(y_0)}\right)^{-N}\int_{B(x_0,2R)}\frac{V(z)}{|x_0-z|^{d-1}}\,dz$$

and

$$III \le \frac{C_N}{|x_0-y_0|t^{d/2}}e^{-\frac{|x_0-y_0|^2}{2t}}\left(1+\frac{\sqrt{t}}{\rho(y_0)}\right)^{-N}.$$

For II we use (5), arriving to

$$\begin{aligned} II &\leq \frac{C_N|x_0-y_0|}{t^{(d+2)/2}} e^{-c\frac{|x_0-y_0|^2}{t}} \left(1+\frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \\ &\leq \frac{C_N}{|x_0-y_0|t^{d/2}} e^{-c\frac{|x_0-y_0|^2}{t}} \left(1+\frac{\sqrt{t}}{\rho(y_0)}\right)^{-N}. \end{aligned}$$

□

We will also need estimates concerning the spatial derivative of the difference between k_t and h_t.

Lemma 5. *Let $V \in \mathrm{RH}_q$ for some $q > d/2$, and $\delta = 2 - d/q$. Then there exist constants c and C such that*

$$|\nabla_1 k_t(x,y) - \nabla_1 h_t(x-y)| \leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \left[\frac{1}{\sqrt{t}} \left(\frac{\sqrt{t}}{|x-y|}\right)^{d-1} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} + G(x,y) \right],$$

for every $|x-y| < \rho(x)$ and $0 < \sqrt{t} < \rho(x)$. Moreover, if $q > d$,

$$|\nabla_1 k_t(x,y) - \nabla_1 h_t(x-y)| \leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{\frac{d+1}{2}}} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta}$$

for every $|x-y| < \rho(x)$ and $t > 0$.

Proof. Applying a perturbation formula as in [6, Eq. (2.10)] (see also [10, Lemma 4.1]) for x, y in $\mathbb{R}^d$ and $t > 0$, we have

$$q_t(x,y) = k_t(x,y) - h_t(x-y) = \int_0^t \int_{\mathbb{R}^d} h_s(x,z) V(z) k_{t-s}(z,y)\, dz dt.$$

So, taking derivatives with respect to x_i,

$$\frac{\partial}{\partial x_i} q_t(x,y) = \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} h_s(x,z) V(z) k_{t-s}(z,y)\, dz dt.$$

Now, we observe that

$$\begin{aligned} \left| \frac{\partial}{\partial x_i} h_s(x,z) \right| &\leq \frac{|x_i - z_i|}{s^{d/2+1}} e^{-\frac{|x-z|^2}{4s}} \\ &= \frac{1}{s^{(d+1)/2}} \frac{|x_i - z_i|}{s^{1/2}} e^{-\frac{|x-z|^2}{4s}} \\ &\leq \frac{C}{s^{(d+1)/2}} e^{-c\frac{|x-z|^2}{s}}. \end{aligned}$$

From the above estimate and (4), we have

$$\begin{aligned} |\nabla q_t(x,y)| &\leq C \left(\int_0^{t/2} + \int_{t/2}^t \right) \int_{\mathbb{R}^d} \frac{e^{-c\frac{|x-z|^2}{s}}}{s^{\frac{d+1}{2}}} V(z) \frac{e^{-c\frac{|y-z|^2}{t-s}}}{(t-s)^{d/2}}\, dz ds \\ &= A + B. \end{aligned}$$

To deal with A observe that $t/2 \leq t-s \leq t$ when $0<s<t/2$, thus

$$\begin{aligned} A &\leq C \int_0^t \int_{\mathbb{R}^d} \frac{e^{-c\frac{|x-z|^2}{s}}}{s^{\frac{d+1}{2}}} V(z) \frac{e^{-c\frac{|y-z|^2}{t}}}{t^{d/2}}\, dzds \\ &= A_1 + A_2, \end{aligned}$$

where A_1 and A_2 are the partition of the inner integral in the regions $B(x, |x-y|/2)$ and its complement, respectively. In A_1 we have $|z-y| \geq |x-y|/2$, hence

$$A_1 \leq \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \int_0^t \int_{B(x,|x-y|/2)} \frac{e^{-c\frac{|x-z|^2}{s}}}{s^{\frac{d+1}{2}}} V(z)\, dzds. \tag{9}$$

Now, setting $\tau = s/|x-y|^2$,

$$\begin{aligned} A_1 &\leq \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \int_{B(x,|x-y|/2)} \frac{V(z)}{|x-z|^{d-1}}\, dz \int_0^\infty \frac{e^{-c/\tau}}{\tau^{\frac{d-1}{2}}} \frac{d\tau}{\tau} \\ &\leq \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \int_{B(x,|x-y|/2)} \frac{V(z)}{|x-z|^{d-1}}\, dz \\ &\leq \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} G(x,y). \end{aligned}$$

Moreover, if $q>d$, starting from (9) after applying Lemma 2 we have

$$\begin{aligned} A_1 &\leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \int_0^t \frac{1}{s^{1/2}} \int_{\mathbb{R}^d} \frac{e^{-c\frac{|x-z|^2}{s}}}{s^{d/2}} V(z)\, dzds \\ &\leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}} \int_0^t \frac{1}{s^{3/2}} \left(\frac{\sqrt{s}}{\rho(x)}\right)^\delta ds \\ &\leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{\frac{d+1}{2}}} \left(\frac{\sqrt{t}}{\rho(x)}\right)^\delta, \end{aligned}$$

where the last integral can be performed since $\delta = 2 - d/q > 1$, in the case $q>d$.

For A_2, using $|z-x| \geq |y-x|/2$, Lemma 2 and the fact that $s<t$ implies $e^{-\frac{|x-y|^2}{s}} \leq e^{-\frac{|x-y|^2}{2s}} e^{-\frac{|x-y|^2}{2t}}$, we have

$$\begin{aligned} A_2 &= C \int_0^t \int_{|z-x|\geq|y-x|/2} \frac{e^{-c\frac{|x-z|^2}{s}}}{s^{\frac{d+1}{2}}} V(z) \frac{e^{-c\frac{|y-z|^2}{t}}}{t^{d/2}}\, dzds \\ &\leq C \left(\int_0^t \frac{e^{-c\frac{|x-y|^2}{s}}}{s^{(d+1)/2}} ds \right) \left(\int_{\mathbb{R}^d} V(z) \frac{e^{-c\frac{|y-z|^2}{t}}}{t^{d/2}}\, dz \right) \\ &\leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t} \left(\frac{\sqrt{t}}{\rho(y)}\right)^\delta \int_0^\infty \frac{e^{-c\frac{|x-y|^2}{s}}}{s^{(d+1)/2}} ds \\ &\leq C \frac{e^{-c\frac{|x-y|^2}{t}}}{t|x-y|^{d-1}} \left(\frac{\sqrt{t}}{\rho(x)}\right)^\delta, \end{aligned}$$

where we have performed a change of variables and that $\rho(y)$ is equivalent to $\rho(x)$.

In the case $q > d$, we can use Lemma 2 before the integration in s to obtain

$$\begin{aligned}
A_2 &\le C\frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}}\int_0^t\int_{|z-x|\ge|y-x|/2}\frac{e^{-c\frac{|x-z|^2}{s}}}{s^{(d+1)/2}}V(z)\,dzds\\
&\le C\frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}}\int_0^t\frac{1}{\sqrt{s}}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-z|^2}{s}}}{s^{d/2}}V(z)\,dzds\\
&\le C\frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/2}}\int_0^t\frac{1}{\sqrt{s}}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta}ds\\
&\le C\frac{e^{-c\frac{|x-y|^2}{t}}}{t^{(d+1)/2}}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta},
\end{aligned}$$

where the last integral can be done since $\delta > 1$.

Now we turn our attention to B. Here $t/2 < s < t$, then

$$\begin{aligned}
B &= \int_{t/2}^t\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-z|^2}{t}}}{t^{(d+1)/2}}V(z)\frac{e^{-c\frac{|y-z|^2}{t-s}}}{(t-s)^{d/2}}\,dzds\\
&= \frac{1}{t^{(d+1)/2}}\int_0^{t/2}\int_{\mathbb{R}^d}e^{-c\frac{|x-z|^2}{t}}V(z)\frac{e^{-c\frac{|y-z|^2}{s}}}{s^{d/2}}\,dzds\\
&= B_1 + B_2,
\end{aligned}$$

where, as before, we define B_1 and B_2 splitting the inner integral in the regions $B(x,|x-y|/2)$ and its complement. In B_1 since $|z-y| \ge |x-y|/2$ and $s < t$ implies $e^{-\frac{|y-z|^2}{s}} \le e^{-\frac{|x-y|^2}{2t}}e^{-\frac{|y-z|^2}{2s}}$, we use Lemma 2 to obtain

$$\begin{aligned}
B_1 &\le C\frac{1}{t^{(d+1)/2}}\int_0^t\int_{|z-y|\ge|x-y|/2}V(z)\frac{e^{-c\frac{|y-z|^2}{s}}}{s^{d/2}}\,dzds\\
&\le C\frac{e^{-c\frac{|x-y|^2}{2t}}}{t^{(d+1)/2}}\int_0^t\int_{\mathbb{R}^d}V(z)\frac{e^{-c\frac{|y-z|^2}{2s}}}{s^{d/2}}\,dzds\\
&\le C\frac{e^{-c\frac{|x-y|^2}{2t}}}{t^{(d+1)/2}}\int_0^t\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(y)}\right)^{\delta}ds\\
&\le C\frac{e^{-c\frac{|x-y|^2}{2t}}}{t^{(d+1)/2}}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta},
\end{aligned}$$

where in the last inequality we have use that $\rho(x) \simeq \rho(y)$ since $|x-y| < \rho(x)$.

Finally, for B_2, we have

$$\begin{aligned}
B_2 &= \frac{1}{t^{(d+1)/2}}\int_0^{t/2}\int_{|x-z|\ge|x-y|/2}e^{-c\frac{|x-z|^2}{t}}V(z)\frac{e^{-c\frac{|y-z|^2}{s}}}{s^{d/2}}\,dzds\\
&\le C\frac{e^{-c\frac{|x-y|^2}{t}}}{t^{(d+1)/2}}\int_0^t\int_{\mathbb{R}^d}V(z)\frac{e^{-c\frac{|y-z|^2}{s}}}{s^{d/2}}\,dzds,
\end{aligned}$$

and we proceed in the same way as in B_1. □

3. $L^p(w)$ inequalities

In this section we are going to give a proof of the $L^p(w)$ boundedness results stated in Theorem 1.

First, we recall the definition of the maximal operators associated to a critical radius function introduced in [2]. Given $\sigma \geq 0$ and $r \geq 1$ we define

$$M_r^{\rho,\sigma} f(x) = \sup_{B(x_0,r_0)\ni x} \left(\fint_{B(x_0,r_0)} |f|^r \right)^{1/r} \left(1 + \frac{r_0}{\rho(x_0)}\right)^{-\sigma},$$

and

$$M_r^{\rho,\mathrm{loc}} f(x) = \sup_{\substack{B(x_0,r_0)\ni x \\ r_0 \leq \rho(x_0)}} \left(\fint_{B(x_0,r_0)} |f|^r \right)^{1/r}.$$

As usual, when $r = 1$ we will simply write $M^{\rho,\sigma}$ and $M^{\rho,\mathrm{loc}}$ respectively. It is clear from the definitions that, for a fixed $r \geq 1$, $M_r^{\rho,\mathrm{loc}} f$ is pointwisely controlled by any $M_r^{\rho,\sigma} f$, $\sigma \geq 0$.

Proposition 1. *Let $1 \leq r < p < \infty$. If a weight $w \in A_{p/r}^{\rho,\infty}$ then there exists $\sigma > 0$ such that $M_r^{\rho,\sigma}$ is bounded on $L^p(w)$. Moreover, if $w^{-r} \in A_1^{\rho,\infty}$ then there exists $\sigma \geq 0$ such that $M_r^{\rho,\sigma}$ is bounded on $L^\infty(w) = \{f : fw \in L^\infty\}$.*

Proof. For $r = 1$ we refer to Proposition 3 in [2]. If $r > 1$ and $w \in A_{p/r}^{\rho,\infty}$ there exists $\sigma > 0$ such that $M^{\rho,\sigma}$ is bounded on $L_w^{p/r}$. Then, for this σ,

$$\begin{aligned}\int [M_r^{\rho,\sigma} f(x)]^p w(x)\,dx &= \int [M^{\rho,\sigma} f^r(x)]^{p/r} w(x)\,dx \\ &\leq C \int |f(x)|^p w(x)\,dx.\end{aligned}$$

Suppose now that $w^{-r} \in A_1^{\rho,\theta}$. For any $B = B(x_0, r)$,

$$\begin{aligned}\left(1 + \frac{r}{\rho(x_0)}\right)^{-\theta} \left(\fint_B |f|^r \right)^{1/r} &\leq \left(1 + \frac{r}{\rho(x_0)}\right)^{-\theta} \|fw\|_\infty \left(\fint_B w^{-r} \right)^{1/r} \\ &\leq C \|fw\|_\infty \inf_B w^{-1}.\end{aligned}$$

Fixing x and taking the supremum over $B \ni x$ we obtain

$$M_r^\theta f(x) \leq C \|fw\|_\infty w^{-1}(x)$$

and the conclusion follows. □

Now we describe some classes of linear operators that can be pointwisely controlled by the maximal operators defined above. Therefore, they will inherit the corresponding $L^p(w)$ boundedness properties.

Lemma 6. *Let $V \in \mathrm{RH}_q$ with $q > d/2$ and $\mathcal{U}$ be an integral operator in L^1_{loc} with kernel $J(x,y)$ such that there exists $\delta > 0$ and for each $N > 0$ there exists a constant $C_N > 0$ such that*

$$|J(x,y)| \le \frac{C_N}{|x-y|^d}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta}. \tag{10}$$

Then, $\mathcal{U}$ is bounded on $L^p(w)$ for $1 \le p < \infty$ and $w \in A_p^{\rho,\infty}$.

Proof. We begin by showing that for each $\theta \ge 0$, there exists a constant $C_\theta > 0$ such that $\mathcal{U}$ satisfy

$$|\mathcal{U}f(x)| \le C_\theta M^{\rho,\theta} f(x),$$

for $f \in L^1_{\mathrm{loc}}$ and $x \in \mathbb{R}^d$. Let $f \in L^1_{\mathrm{loc}}$ and $x \in \mathbb{R}^d$. We split

$$\begin{aligned}|\mathcal{U}f(x)| &\le \int_{B(x,\rho(x))} |J(x,y)||f(y)|\,dy + \int_{B(x,\rho(x))^c} |J(x,y)||f(y)|\,dy\\ &= A(x) + B(x).\end{aligned}$$

For A, applying the estimate (10), splitting in dyadic annuli and setting $B^k_\rho = B(x, 2^k\rho(x))$,

$$\begin{aligned}A(x) &\le \frac{C}{\rho(x)^\delta}\int_{B(x,\rho(x))}\frac{|f(y)|}{|x-y|^{d-\delta}}\\ &\le \frac{C}{\rho(x)^\delta}\sum_{k=-\infty}^{0}(2^k\rho(x))^{\delta-d}\int_{B^k_\rho}|f(y)|\,dy\\ &\le C\sum_{-\infty}^{0}2^{k\delta}\fint_{B^k_\rho}|f(y)|\,dy\\ &\le CM^{\rho,\mathrm{loc}}f(x).\end{aligned}$$

Similarly, for any $\theta \ge 0$,

$$\begin{aligned}B(x) &\le C_N\int_{B(x,\rho(x))^c}\frac{|f(y)|}{|x-y|^d}\left(\frac{\rho(x)}{|x-y|}\right)^{N-\delta}dy\\ &\le C_N\sum_{k=0}^{\infty}\frac{2^{k(\delta-N)}}{(2^k\rho(x))^d}\int_{B^k_\rho}|f(y)|\,dy\\ &\le C_N M^{\rho,\theta}f(x)\sum_{k=0}^{\infty}2^{k(\delta+\theta-N)}\\ &\le C_\theta M^{\rho,\theta}f(x),\end{aligned}$$

choosing $N > \theta + \delta$. Therefore, $\mathcal{U}$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p^{\rho,\infty}$ by applying Proposition 1.

Next, we show that its adjoing operator, $\mathcal{U}^\star$ is also pointwise controlled by the operator $M^{\rho,\theta}$ for $f \in L^1_{\mathrm{loc}}$ and $x \in \mathbb{R}^d$. Applying (2), for each $N > 0$ there exists

C_N such that,

$$|J(x,y)| \leq \frac{C_N}{|x-y|^d}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta}$$
$$\leq \frac{C_N}{|x-y|^d}\left(1+\frac{|x-y|}{\rho(y)}\right)^{-N+N_0+2}\left(\frac{|x-y|}{\rho(y)}\right)^{\delta}.$$

Finally, we will show that this pointwise control of $\mathcal{U}^\star$ guarantees the boundedness on $L^1(w)$ for $\mathcal{U}$ as long as $w \in A_1^{\rho,\infty}$. Let $w \in A_1^{\rho,\infty}$. Applying Proposition 1,

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{U}(f)g &= \int_{\mathbb{R}^d} f\,\mathcal{U}^\star(g)\\ &\leq C\|\mathcal{U}^\star(g)w^{-1}\|_\infty \|fw\|_1\\ &\leq C\|gw^{-1}\|_\infty\|fw\|_1.\end{aligned}$$

Therefore

$$\begin{aligned}\|\mathcal{U}(f)w\|_1 &= \sup_{\|h\|_\infty=1}\int_{\mathbb{R}^d}\mathcal{U}(f)wh\\ &= \sup_{\|gw^{-1}\|_\infty=1}\int_{\mathbb{R}^d}\mathcal{U}(f)g\\ &= \sup_{\|gw^{-1}\|_\infty=1}\int_{\mathbb{R}^d} f\,\mathcal{U}^\star(g)\\ &\leq C\|fw\|_1.\end{aligned}$$

□

Lemma 7. *Let $V \in \mathrm{RH}_q$ with $q > d/2$ and $\mathcal{U}$ be an integral operator with kernel $J(x,y)$ such that for each $N > 0$ there exists a constant $C_N > 0$ such that*

$$|J(x,y)| \leq \frac{C_N}{|x-y|^{d-1}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} G(x,y), \tag{11}$$

with G defined in (6). *Let $1/\nu = (1/q-1/d)^+$. Then, $\mathcal{U}$ is bounded on $L^p(w)$ for $1 < p < \nu$ with w such that $w^{-1/(p-1)} \in A^{\rho,\infty}_{p'/\nu'}$ and it is also bounded in $L^1(w)$ for w such that $w^{\nu'} \in A_1^{\rho,\infty}$.*

Proof. Let us first observe that if $q > d$ we can apply estimate (7) to obtain, for each $N > 0$,

$$|J(x,y)| \leq \frac{C_N}{|x-y|^{d-1}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} G(x,y)$$
$$\leq \frac{C_N}{|x-y|^d}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N+N_2}\left(\frac{|x-y|}{\rho(x)}\right)^{\delta}.$$

Therefore, we can apply Lemma 6 to obtain the stated conclusions.

The openness of reverse-Hölder classes implies that it only remains to consider the case $d/2 < q < d$. We will show that for each $\theta \geq 0$ there exists a constant $C_\theta > 0$ such that its adjoint $\mathcal{U}^\star$ satisfies

$$|\mathcal{U}^\star f(x)| \leq C_\theta M^\theta_{\nu'} f(x)$$

for $f \in L^1_{\text{loc}}$ and $x \in \mathbb{R}^d$, where $1/\nu = (1/q - 1/d)^+$. Let $f \in L^1_{\text{loc}}$ and $x \in \mathbb{R}^d$. We split

$$\begin{aligned}|\mathcal{U}^\star f(x)| &\leq \int_{B(x,\rho(x))} |J(y,x)||f(y)|\,dy + \int_{B(x,\rho(x))^c} |J(y,x)||f(y)|\,dy \\ &= A(x) + B(x).\end{aligned}$$

For A, applying the estimate (11) and splitting in dyadic annuli and setting $B^k_\rho = B(x, 2^k\rho(x))$,

$$\begin{aligned}A(x) &\leq C \int_{B(x,\rho(x))} \frac{|f(y)|}{|x-y|^{d-1}} \int_{B(y,|x-y|)} \frac{V(z)}{|y-z|^{d-1}}\,dzdy \\ &\leq C \int_{B(x,\rho(x))} \frac{|f(y)|}{|x-y|^{d-1}} \int_{B(x,2|x-y|)} \frac{V(z)}{|y-z|^{d-1}}\,dzdy \\ &\leq C \sum_{k=-\infty}^{0} (2^k\rho(x))^{1-d} \int_{B^k_\rho} |f(y)| \int_{B^k_\rho} \frac{V(z)}{|y-z|^{d-1}}\,dzdy. \\ &\leq C \sum_{k=-\infty}^{0} (2^k\rho(x))^{1-d} \int_{B^k_\rho} |f(y)||I_1(V\chi_{B^k_\rho})(y)|\,dy \\ &\leq C \sum_{k=-\infty}^{0} (2^k\rho(x))^{1-d+\frac{d}{\nu'}} \left(\fint_{B^k_\rho} |f(y)|^{\nu'}\,dy \right)^{1/\nu'} \left(\int_{B^k_\rho} |I_1(V\chi_{B^k_\rho})(y)|^{\nu}\,dy \right)^{1/\nu} dy,\end{aligned}$$

where I_1 stands for the classical fractional integral operator which is bounded from L^q into L^ν. So,

$$\begin{aligned}A(x) &\leq CM^{\rho,\text{loc}}_{\nu'}f(x) \sum_{k=-\infty}^{0} (2^k\rho(x))^{1-d+\frac{d}{\nu'}+\frac{d}{q}} \left(\fint_{B^k_\rho} V^q(y)\,dy \right)^{1/q} \\ &\leq CM^{\rho,\text{loc}}_{\nu'}f(x) \sum_{k=-\infty}^{0} (2^k\rho(x))^2 \fint_{B^k_\rho} V(z)\,dz \\ &\leq CM^{\rho,\text{loc}}_{\nu'}f(x) \sum_{k=-\infty}^{0} 2^{k\delta} \leq CM^{\rho,\text{loc}}f(x),\end{aligned}$$

where we have used again Lemma 1.

Now we turn our attention to $B(x)$. Applying estimate (11), splitting again in dyadic annuli we proceed as above to obtain

$$\begin{aligned}B(x) &\leq C_N \int_{B(x,\rho(x))^c} \frac{|f(y)|}{|x-y|^{d-1}} \left(\frac{\rho(x)}{|x-y|}\right)^N \int_{B(y,|x-y|)} \frac{V(z)}{|y-z|^{d-1}}\,dzdy \\ &\leq C_N \int_{B(x,\rho(x))^c} \frac{|f(y)|}{|x-y|^{d-1}} \left(\frac{\rho(x)}{|x-y|}\right)^N \int_{B(x,2|x-y|)} \frac{V(z)}{|y-z|^{d-1}}\,dzdy \\ &\leq C_N \sum_{k=0}^{\infty} (2^k\rho(x))^{1-d} 2^{-kN} \int_{B^k_\rho} |f(y)| \int_{B^k_\rho} \frac{V(z)}{|y-z|^{d-1}}\,dzdy.\end{aligned}$$

$$\begin{aligned}
&\leq C_N \sum_{k=0}^{\infty} (2^k\rho(x))^{1-d} 2^{-kN} \int_{B_\rho^k} |f(y)||I_1(V\chi_{B_\rho^k})(y)|\,dy \\
&\leq C_N \sum_{k=0}^{\infty} (2^k\rho(x))^{1-d+\frac{d}{\nu'}} 2^{-kN} \left(\fint_{B_\rho^k} |f(y)|^{\nu'}\,dy \right)^{1/\nu'} \\
&\qquad\qquad\qquad\qquad \times \left(\int_{B_\rho^k} |I_1(V\chi_{B_\rho^k})(y)|^{\nu}\,dy \right)^{1/\nu} dy \\
&\leq C_N M_{\nu'}^{\rho,\theta} f(x) \sum_{k=0}^{\infty} (2^k\rho(x))^{1-d+\frac{d}{\nu'}+\frac{d}{q}} 2^{-k(N-\theta)} \left(\fint_{B_\rho^k} V^q(y)\,dy \right)^{1/q} \\
&\leq C_N M_{\nu'}^{\rho,\theta} f(x) \sum_{k=0}^{\infty} (2^k\rho(x))^{2} 2^{-k(N-\theta)} \fint_{B_\rho^k} V(z)\,dz \\
&\leq C_N M_{\nu'}^{\rho,\theta} f(x) \sum_{k=0}^{\infty} 2^{-k(N-\theta-N_2)} \leq C_\theta M_{\nu'}^{\rho,\theta} f(x),
\end{aligned}$$

where we have used again Lemma 1.

Finally, proceeding as in the proof of Lemma 6, we derive the boundedness properties in $L^p(w)$ follow from the ones of $M_{\nu'}^{\rho,\theta}$ stated in Proposition 1. ☐

To prove Theorem 1 we are going to use a well known technique to deal with Schrödinger type operators that involve adding and subtracting (at a local scale) the corresponding classical operator. Let us consider

$$\mathcal{H}_\Delta f(x) = \left(\int_0^\infty \left[\int_{\mathbb{R}^d} \nabla_1 h_t(x-y) f(y)\,dy \right]^2 dt \right)^{1/2},$$

where h_t is the classical kernel defined in (3) It is known from the vector valued Calderón–Zygmund theory that $\mathcal{H}_\Delta$ is a bounded operator in $L^p(w)$ for $1<p<\infty$ and $w \in A_p$ and is of weak type $(1,1)$ for $w \in A_1$.

In what follows, we need to consider the operators

$$\mathcal{H}_\Delta^{\mathrm{loc}} f(x) = \left(\int_0^\infty \left[\int_{|x-y|<\rho(x)} \nabla_1 h_t(x-y) f(y)\,dy \right]^2 dt \right)^{1/2}$$

and

$$\mathcal{H}_\Delta^0 f(x) = \sum_{j\in\mathbb{N}} \chi_{Q_j} |\mathcal{H}_\Delta(f\chi_{\widetilde{Q_j}})|, \tag{12}$$

where $\{Q_j\}_{j\in\mathbb{N}}$ is a covering of critical balls such that the family of a fixed dilation of them, $\{\widetilde{Q_j}\}_{j\in\mathbb{N}}$ has bounded overlapping, as the one given in Lemma 3. According to Proposition 1 in [3], the boundedness properties of $\mathcal{H}_\Delta$ imply that $\mathcal{H}_\Delta^0$ is bounded in $L^p(w)$ for $1<p<\infty$ and $w \in A_p^{\rho,\mathrm{loc}}$ and of weak type $(1,1)$ for $w \in A_1^{\rho,\mathrm{loc}}$.

We are now in a position to give a proof of Theorem 1.

Proof of Theorem 1. We begin by decomposing

$$
\begin{aligned}
\mathcal{H}_S f(x) &= \left(\int_0^\infty \left[\int_{\mathbb{R}^d} \nabla_1 k_t(x,y) f(y)\,dy\right]^2 dt\right)^{1/2} \\
&\leq \left(\int_0^{\rho(x)^2} \left[\int_{|x-y|<\rho(x)} |\nabla_1 k_t(x,y) - \nabla_1 h_t(x-y)||f(y)|\,dy\right]^2 dt\right)^{1/2} \\
&\quad + \left(\int_0^{\rho(x)^2} \left[\int_{|x-y|<\rho(x)} \nabla_1 h_t(x-y) f(y)\,dy\right]^2 dt\right)^{1/2} \\
&\quad + \left(\int_{\rho(x)^2}^\infty \left[\int_{|x-y|<\rho(x)} |\nabla_1 k_t(x,y)||f(y)|\,dy\right]^2 dt\right)^{1/2} \\
&\quad + \left(\int_0^\infty \left[\int_{|x-y|\geq\rho(x)} |\nabla_1 k_t(x,y)||f(y)|\,dy\right]^2 dt\right)^{1/2} \\
&= I + II + III + IV.
\end{aligned}
$$

First, we are going to check that terms I, III and IV above are bounded by operators whose kernels satisfy the hypothesis of either Lemma 6 or Lemma 7. In any case, an application of such lemma lead to the conclusions of the theorem.

We deal first with the term I. We apply the Minkowski inequality to obtain

$$
I \leq \int_{|x-y|<\rho(x)} \left(\int_0^{\rho(x)^2} |\nabla_1 k_t(x,y) - \nabla_1 h_t(x-y)|^2 dt\right)^{1/2} |f(y)|\,dy.
$$

Suppose first that $q<d$, using Lemma 5, we obtain

$$
\begin{aligned}
I &\leq C\int_{|x-y|<\rho(x)} |x-y|^{1-d}\rho(x)^{-\delta} \left(\int_0^{\rho(x)^2} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/q}} dt\right)^{1/2} |f(y)|\,dy \\
&\quad + C\int_{|x-y|<\rho(x)} G(x,y) \left(\int_0^{\rho(x)^2} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d}} dt\right)^{1/2} |f(y)|\,dy \\
&\leq I_1 + I_2.
\end{aligned}
$$

Observe that

$$
\left(\int_0^{\rho(x)^2} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d/q}} dt\right)^{1/2} \leq |x-y|^{1-d/q}.
$$

Then

$$I_1 \leq C \int \frac{\chi_{B(x,\rho(x))}}{|x-y|^d} \left(\frac{|x-y|}{\rho(x)}\right)^{2-d/q} |f(y)|\, dy.$$

As for I_2, we have

$$\left(\int_0^{\rho(x)^2} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^d} dt\right)^{1/2} \leq |x-y|^{1-d},$$

and

$$I_2 \leq C \int \chi_{B(x,\rho(x))} \frac{G(x,y)}{|x-y|^{d-1}} |f(y)|\, dy.$$

Therefore I is bounded by a sum of two operators whose kernels satisfy (10) or (11) and we can apply Lemma 6 and Lemma 7.

On the other hand, if $q > d$ then

$$I \leq \rho(x)^{d/q-2} \int_{|x-y|<\rho(x)} \left(\int_0^{\rho(x)^2} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d-1+d/q}} dt\right)^{1/2} |f(y)|\, dy,$$

$$\left(\int_0^{\rho(x)^2} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{d-1+d/q}} dt\right)^{1/2} \leq |x-y|^{2-d-d/q}.$$

So

$$I \leq C \int \frac{\chi_{B(x,\rho(x))}}{|x-y|^{d-1}\rho(x)} |f(y)|\, dy.$$

In this case, I is bounded by an operator whose kernel satisfies the hypothesis of Lemma 6.

For III we will apply the Minkowski inequality and Lemma 4 to obtain

$$\begin{aligned} III &\leq \int_{|x-y|<\rho(x)} \left(\int_{\rho(x)^2}^{\infty} |\nabla_1 k_t(x,y)|^2 dt\right)^{1/2} |f(y)|\, dy \\ &\leq C \int_{|x-y|<\rho(x)} \left(\int_{\rho(x)^2}^{\infty} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^d} dt\right)^{1/2} \left(\frac{1}{|x-y|} + G(x,y)\right) |f(y)|\, dy. \end{aligned}$$

Now, setting $\tau = t/|x-y|^2$, we compute

$$\int_{\rho(x)^2}^{\infty} \frac{e^{-c\frac{|x-y|^2}{t}}}{t^d} dt \leq C|x-y|^{-2(d-1)} \int_{\rho(x)^2/|x-y|^2}^{\infty} \frac{d\tau}{\tau^d} \leq C\rho(x)^{-2d+2}.$$

Therefore, we can bound

$$\begin{aligned} III &\leq C \int_{|x-y|<\rho(x)} \left(\frac{1}{|x-y|\rho(x)^{d-1}} + \frac{G(x,y)}{|x-y|^{d-1}} \right) |f(y)|\, dy \\ &\leq C_N \int_{\mathbb{R}^d} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N} \left(\frac{|x-y|}{\rho(x)}\right)^{d-1} \frac{|f(y)|}{|x-y|^d}\, dy \\ &\qquad + C_N \int_{\mathbb{R}^d} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N} \frac{G(x,y)}{|x-y|^{d-1}} |f(y)|\, dy. \end{aligned}$$

Then, as before, we see that III is bounded by a sum of two operators whose kernels satisfy (10) or (11).

To deal with IV we apply again the Minkowski inequality and Lemma 4 to obtain

$$\begin{aligned} IV &\leq \int_{|x-y|\geq\rho(x)} \left(\int_0^\infty |\nabla_1 k_t(x,y)|^2 dt \right)^{1/2} |f(y)|\, dy \\ &\leq C_N \int_{|x-y|\geq\rho(x)} \left(\int_0^\infty \frac{e^{-c\frac{|x-y|^2}{t}}}{t^d} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-2N} dt \right)^{1/2} \\ &\qquad\qquad \times \left(\frac{1}{|x-y|} + G(x,y) \right) |f(y)|\, dy. \end{aligned}$$

Now, setting $\tau = |x-y|^2/t$,

$$\begin{aligned} &\int_0^\infty \frac{e^{-c\frac{|x-y|^2}{t}}}{t^d} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-2N} dt \\ &\quad = |x-y|^{-2(d-1)} \int_0^\infty e^{-c\tau} \tau^{d-1} \left(1 + \frac{|x-y|}{\sqrt{\tau}\rho(y)}\right)^{-2N} \frac{d\tau}{\tau} \\ &\quad \leq |x-y|^{-2(d-1)} \left[\left(\frac{|x-y|}{\rho(y)}\right)^{-2N} \int_0^{\frac{|x-y|^2}{\rho(y)^2}} e^{-c\tau} \tau^{d-2+N} d\tau \right. \\ &\qquad\qquad \left. + \int_{\frac{|x-y|^2}{\rho(y)^2}}^\infty e^{-c\tau} \tau^{d-2} d\tau \right] \qquad (13) \\ &\quad \leq |x-y|^{-2(d-1)} \left(\frac{|x-y|}{\rho(y)}\right)^{-2N} \left[\int_0^{\frac{|x-y|^2}{\rho(y)^2}} e^{-c\tau} \tau^{d-2+N} d\tau \right. \\ &\qquad\qquad \left. + \int_{\frac{|x-y|^2}{\rho(y)^2}}^\infty e^{-c\tau} \tau^{d-2+N} d\tau \right] \\ &\quad \leq C_N |x-y|^{-2(d-1)} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-2N}. \end{aligned}$$

Altogether, we obtain

$$\begin{aligned} IV &\leq C_N \int_{|x-y|\geq\rho(x)} \left(1+\frac{|x-y|}{\rho(y)}\right)^{-N} \left[\frac{1}{|x-y|^d}+\frac{G(x,y)}{|x-y|^{d-1}}\right] |f(y)|\,dy \\ &\leq C_N \int \left(1+\frac{|x-y|}{\rho(y)}\right)^{-N} \left(\frac{|x-y|}{\rho(x)}\right)^{\delta} \frac{|f(y)|}{|x-y|^d}\,dy \\ &+ C_N \int \left(1+\frac{|x-y|}{\rho(y)}\right)^{-N} \frac{G(x,y)}{|x-y|^{d-1}}|f(y)|\,dy, \end{aligned}$$

and we conclude as was done for III.

To finish the proof we need to take care of

$$II = \left(\int_0^{\rho(x)^2} \left[\int_{|x-y|<\rho(x)} \nabla_1 h_t(x-y) f(y)\,dy\right]^2 dt\right)^{1/2} = \mathcal{H}_\Delta^{\mathrm{loc}} f(x).$$

We will do so by comparing it with $\mathcal{H}_\Delta^0$ defined in (12) which is bounded on $L^p(w)$ for $1<p<\infty$ and of weak type $(1,1)$ as long as $w\in A_p^{\rho,\mathrm{loc}}\supset A_p^{\rho,\infty}$.

Let $\{Q_j\}_{j\in\mathbb{N}}$ be the covering given in Lemma 3. First, we observe that for $x\in Q_j$,

$$|\mathcal{H}_\Delta(f\chi_{\widetilde{Q_j}})(x)-\mathcal{H}_\Delta^{\mathrm{loc}} f(x)| \leq \left(\int_0^\infty \left|\int_{\widetilde{Q_j}\setminus B(x,\rho(x))} \nabla_1 h_t(x-y) f(y)\,dy\right|^2 dt\right)^{1/2}.$$

If $y\in\widetilde{Q_j}\setminus B(x,\rho(x))$ there exist c_1 and c_2 such that

$$c_1\rho(x_j)\leq\rho(x)\leq|x-y|\leq c_2\rho(x_j).$$

So,

$$\begin{aligned} |\mathcal{H}_\Delta(f\chi_{\widetilde{Q_j}})(x)-\mathcal{H}_\Delta^{\mathrm{loc}} f(x)| &\leq C\left(\int_0^\infty \frac{e^{-c\rho(x_j)^2/t}}{t^{d+2}}\rho(x_j)^2 dt\right)^{1/2} \int_{\widetilde{Q_j}} |f(y)|\,dy \\ &\leq C\left(\int_0^\infty \frac{e^{-c\rho(x_j)^2/t}}{t^{d+1}} dt\right)^{1/2} \rho(x_j)^d M^{\rho,\mathrm{loc}} f(x) \\ &\leq C M^{\rho,\mathrm{loc}} f(x). \end{aligned} \tag{14}$$

Finally, for $1<p<\infty$,

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{H}_\Delta^{\mathrm{loc}} f(x)|^p w(x)\,dx &\leq \sum_{j\in\mathbb{N}} \int_{Q_j} |\mathcal{H}_\Delta(f\chi_{B(x,\rho(x))})(x)-\mathcal{H}_\Delta(f\chi_{\widetilde{Q_j}})(x)|^p w(x)\,dx \\ &+ \sum_{j\in\mathbb{N}} \int_{Q_j} |\mathcal{H}_\Delta(f\chi_{\widetilde{Q_j}})(x)|^p w(x)\,dx \\ &= A+B. \end{aligned}$$

To deal with A we recall that $\sum_{j\in\mathbb{N}} \chi_{\widetilde{Q_j}} \leq N_1$. This and (14) give

$$A \leq \int \sum_{j\in\mathbb{N}} \chi_{Q_j} |M^{\rho,\mathrm{loc}} f(x)|^p w(x)\, dx \leq C \int |M^{\rho,\mathrm{loc}} f(x)|^p w(x)\, dx,$$

and the result follows since $M^{\rho,\mathrm{loc}}$ is bounded on $L^p(w)$ for $w \in A_p^{\rho,\mathrm{loc}} \supset A_p^{\rho,\infty}$.

To estimate B we proceed in a similar way to obtain

$$\begin{aligned} B &\leq \int \sum_{j\in\mathbb{N}} \chi_{Q_j} |\mathcal{H}_\Delta (f \chi_{\widetilde{Q_j}}(x)|^p w(x)\, dx \\ &\leq C \int \sum_{j\in\mathbb{N}} \chi_{Q_j} |\sum_{j\in\mathbb{N}} \chi_{Q_j} \mathcal{H}_\Delta (f \chi_{\widetilde{Q_j}}(x)|^p w(x)\, dx \\ &\leq C \int |\mathcal{H}_\Delta^0 f(x)|^p w(x)\, dx, \end{aligned}$$

and the result follows since now $\mathcal{H}_\Delta^0$ is bounded on $L^p(w)$ for $w \in A_p^{\rho,\mathrm{loc}} \supset A_p^{\rho,\infty}$.

When $p = 1$ we proceed in a similar way using again (14) and that $\mathcal{H}_\Delta^0$ is of weak type $(1,1)$ with respect to w for $w \in A_1^{\rho,\mathrm{loc}} \supset A_1^{\rho,\infty}$. □

To finish this section we prove Theorem 2.

Proof. First, we apply the Minkowski inequality to estimate

$$\begin{aligned} \mathcal{H}_V f(x) &= \left(\int_0^\infty \left(\int_{\mathbb{R}^d} V^{1/2}(x) k_t(x,y) f(y)\, dy \right)^2 dt \right)^{1/2} \\ &\leq V^{1/2}(x) \int_{\mathbb{R}^d} \left(\int_0^\infty k_t(x,y)^2 dt \right)^{1/2} f(y)\, dy. \end{aligned}$$

Now, we apply inequality (4) to bound the inner integral in the following way:

$$\begin{aligned} \left(\int_0^\infty k_t(x,y)^2 dt \right)^{1/2} &\leq C_N \left(\int_0^\infty \frac{e^{-c\frac{|x-y|^2}{t}}}{t^d} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-2N} dt \right)^{1/2} \\ &\leq \frac{C_N}{|x-y|^{d-1}} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N}, \end{aligned}$$

as in (13).

So, $\mathcal{H}_V$ is pointwise bounded by an integral operator $\mathcal{U}$ with kernel $J(x,y)$ satisfying, for each $N > 0$,

$$|J(x,y)| \leq C_N \frac{V^{1/2}(x)}{|x-y|^{d-1}} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N}.$$

The proof will end by showing that for each $\theta \geq 0$ there exists a constant C_θ such that its adjoint $\mathcal{U}^\star$ satisfies

$$|\mathcal{U}^\star f(x)| \leq C_\theta M^\theta_{(2q)'} f(x)$$

for $f \in L^1_{\rm loc}$ and $x \in \mathbb{R}^d$. Consequently $\mathcal{U}$ is bounded on $L^p(w)$ for $1 < p < 2q$ with w such that $w^{-1/(p-1)} \in A^{\rho,\infty}_{p'/(2q)'}$ and it is also bounded in $L^1(w)$ for w such that $w^{(2q)'} \in A^{\rho,\infty}_1$.

For a fixed $x \in \mathbb{R}^d$ let $B_\rho = B(x, \rho(x))$. First we decompose

$$\begin{aligned} |\mathcal{U}^\star f(x)| &\le C \int_{B_\rho} \frac{V^{1/2}(y) f(y)}{|x-y|^{d-1}}\, dy + C_N \int_{(B_\rho)^c} \frac{V^{1/2}(y) f(y)}{|x-y|^{d-1}} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N} dy \\ &= A + B. \end{aligned}$$

Now, let $B^k_\rho = 2^k B_\rho$ for $k \in \mathbb{Z}$. For A we use Hölder's inequality, the reverse-Hölder condition for V and Lemma 1 to obtain

$$\begin{aligned} A &\le C \sum_{k=-\infty}^{0} \frac{1}{(2^k \rho(x))^{d-1}} \int_{B^k_\rho \setminus B^{k-1}_\rho} V^{1/2}(y) f(y)\, dy \\ &\le C \sum_{k=-\infty}^{0} \frac{1}{(2^k \rho(x))^{d-1}} \left(\int_{B^k_\rho} V^q(y)\, dy \right)^{1/(2q)} \left(\int_{B^k_\rho} f^{(2q)'}(y)\, dy \right)^{1/(2q)'} \\ &= C \sum_{k=-\infty}^{0} 2^k \rho(x) \left(\fint_{B^k_\rho} V^q(y)\, dy \right)^{1/(2q)} \left(\fint_{B^k_\rho} f^{(2q)'}(y)\, dy \right)^{1/(2q)'} \\ &\le C M^{\rho,\rm loc}_{(2q)'} f(x) \sum_{k=-\infty}^{0} 2^k \rho(x) \left(\fint_{B^k_\rho} V(y)\, dy \right)^{1/2} \\ &\le C M^{\rho,\rm loc}_{(2q)'} f(x) \sum_{k=-\infty}^{0} 2^{k(1-d/(2q))} \le C M^{\rho,\rm loc}_{(2q)'} f(x), \end{aligned}$$

since $1 - d/(2q) > 0$.

To deal with B, use similar arguments to get, for any $\theta \ge 0$,

$$\begin{aligned} B &\le C_N \sum_{k=0}^{\infty} \frac{2^{-kN}}{(2^k \rho(x))^{d-1}} \int_{B^k_\rho \setminus B^{k-1}_\rho} V^{1/2}(y) f(y)\, dy \\ &\le C_N \sum_{k=0}^{\infty} 2^k \rho(x) 2^{-k(N-\theta)} \left(\fint_{B^k_\rho} V^q(y)\, dy \right)^{1/(2q)} 2^{-k\theta} \left(\fint_{B^k_\rho} f^{(2q)'}(y)\, dy \right)^{1/(2q)'} \\ &\le C_N M^{\rho,\theta}_{(2q)'} f(x) \sum_{k=0}^{\infty} 2^k \rho(x) 2^{-k(N-\theta)} \left(\fint_{B^k_\rho} V(y)\, dy \right)^{1/2} \\ &\le C_N M^{\rho,\theta}_{(2q)'} f(x) \sum_{k=-\infty}^{0} 2^{k(N-\theta-N_2/2)} \le C_\theta M^{\rho,\theta}_{(2q)'} f(x), \end{aligned}$$

choosing $N > \theta + N_2/2$. □

4. Littlewood–Paley function associated to $\mathcal{L}^\alpha$

In classical Harmonic Analysis, the usual Littlewood–Paley function is defined in terms of the Poisson kernel, that is, that associated to the semigroup $e^{-t\sqrt{-\Delta}}$. In the Schrödinger context, when taking spatial derivatives, we have the following definition:

$$\mathcal{G}_S f(x) = \left(\int_0^\infty \left| \nabla e^{-t\sqrt{\mathcal{L}}} f(x) \right|^2 t\, dt \right)^{1/2}.$$

This is a special case of the family of square functions: for $0 < \alpha \leq 1$ we set

$$\mathcal{H}_S^\alpha f(x) = \left(\int_0^\infty \left| \nabla e^{-t\mathcal{L}^\alpha} f(x) \right|^2 t^{1/\alpha} \frac{dt}{t} \right)^{1/2}.$$

In fact, taking $\alpha = 1/2$ we recover $\mathcal{G}_S$ and moreover the square function $\mathcal{H}_S$ coincides with $\mathcal{H}_S^1$.

Since the kernel of the semigroup $e^{-t\mathcal{L}^\alpha}$ can be written in terms of the kernel of $e^{-t\mathcal{L}}$, we will show that the boundedness properties shown for the case $\alpha = 1$ can be transferred to the other values of α. In particular we obtain boundedness properties for the Poisson case.

Following [12, Section 3.2] (see also [7] and [13, Chapter IX, Section 11]) the kernel of the semigroup generated by $\mathcal{L}^\alpha$, say k_t^α, is defined through the subordination formula in terms of k_t taking the form

$$k_t^\alpha(x, y) = \int_0^\infty \eta_t^\alpha(s)\, k_s(x, y)\, ds,$$

where $\eta_t^\alpha(s) = \frac{1}{t^{1/\alpha}} \eta_1^\alpha(\frac{s}{t^{1/\alpha}})$ and moreover it satisfies

$$\int_0^\infty s^{-\gamma}\, \eta_1^\alpha(s) ds < \infty, \tag{15}$$

for any $\gamma > 0$. Therefore, performing the change of variable $u = t^{-1/\alpha}s$,

$$k_t^\alpha(x, y) = \int_0^\infty \eta_1^\alpha(u)\, k_{t^{1/\alpha}u}(x, y)\, du.$$

Consequently, if we call $B_\alpha = L^2((0,\infty), t^{1/\alpha}\frac{dt}{t})$,

$$\mathcal{H}_S^\alpha f(x) = \left\| \int_0^\infty \eta_1^\alpha(u) \int_{\mathbb{R}^d} \nabla k_{t^{1/\alpha}u}(x, y)\, f(y)\, dy\, du \right\|_{B_\alpha}.$$

Applying the Minkowski integral inequality and changing now the variable t by $s = t^{1/\alpha}u$ we arrive to

$$\mathcal{H}_S^\alpha f(x) \leq \int_0^\infty \frac{\eta_1^\alpha(u)}{u^{1/2}} \left\| \int_{\mathbb{R}^d} \nabla k_s(x, y)\, f(y)\, dy \right\|_{B_1} du,$$

and using (15) we may conclude

$$\mathcal{H}_S^\alpha f(x) \leq C \mathcal{H}_S f(x). \tag{16}$$

From the above inequality we derive the weighted L^p boundedness of Theorem 1 for $\mathcal{H}_S^\alpha$ for any $0 < \alpha < 1$.

For the square functions involving V we consider the family of functions for $0<\alpha\le 1$

$$\mathcal{H}_V^\alpha f(x)=\left(\int_0^\infty \left|V^{1/2}e^{-tL^\alpha}f(x)\right|^2 t^{1/\alpha}\frac{dt}{t}\right)^{1/2}.$$

Again, taking $\alpha=1/2$ we recover $\mathcal{G}_V$ and $\mathcal{H}_V$ coincides with $\mathcal{H}_V^1$.

Proceeding as above we obtain

$$\mathcal{H}_V^\alpha f(x)\le C\mathcal{H}_V f(x). \tag{17}$$

From (16), (17), Theorem 1 and Theorem 2 we can derive the following boudnedness results for $\mathcal{H}_S^\alpha$ and $\mathcal{H}_V^\alpha$.

Theorem 3. *Let $V\in \mathrm{RH}_q$ for $q>d/2$ and $d\ge 3$. For $0<\alpha\le 1$, $\mathcal{H}_S^\alpha$ is bounded on $L^p(w)$ for $1<p<p_0$ with p_0 such that $\frac{1}{p_0}=\left(\frac{1}{q}-\frac{1}{d}\right)^+$ and w such that $w^{-\frac{1}{p-1}}\in A^\rho_{p'/p_0'}$. Moreover it is of weak type $(1,1)$ for weights satisfying $w^{p_0'}\in A_1^\rho$.*

Theorem 4. *Let $V\in \mathrm{RH}_q$ for $q>d/2$ and $d\ge 3$. For $0<\alpha\le 1$, $\mathcal{H}_V^\alpha$ is bounded on $L^p(w)$ for $1<p<2q$ with w such that $w^{-1/(p-1)}\in A^{\rho,\infty}_{p'/(2q)'}$, and it is also bounded in $L^1(w)$ for w such that $w^{(2q)'}\in A_1^{\rho,\infty}$.*

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