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GENUS AND BOOK THICKNESS OF REDUCED COZERO-DIVISOR GRAPHS OF COMMUTATIVE RINGS

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ABSTRACT. For a commutative ring R with identity, let $\langle a\rangle$ be the principal ideal generated by $a \in R$. Let $\Omega(R)^*$ be the set of all nonzero proper principal ideals of R. The reduced cozero-divisor graph $\Gamma_r(R)$ of R is the simple undirected graph whose vertex set is $\Omega(R)^*$ and such that two distinct vertices $\langle a\rangle$ and $\langle b\rangle$ in $\Omega(R)^*$ are adjacent if and only if $\langle a\rangle \nsubseteq \langle b\rangle$ and $\langle b\rangle \nsubseteq \langle a\rangle$. In this article, we study certain properties of embeddings of the reduced cozero-divisor graph of commutative rings. More specifically, we characterize all Artinian nonlocal rings whose reduced cozero-divisor graph has genus two. Also we find the book thickness of the reduced cozero-divisor graphs which have genus at most one.

1. Introduction

Throughout this paper R is a commutative Artinian nonlocal ring with identity. The study of algebraic graph theory deepens our understanding of the connections between algebra and graph theory. In 1988, Beck [6] took the first step by introducing the idea of linking a commutative ring to its associated graph. Afterwards, Anderson and Livingston [3] modified the definition given by Beck [6] and studied the graph as the zero-divisor graph of commutative rings. Motivated by their works, many researchers introduced and studied several other graphs related to commutative rings with identity. One can refer to [2] for the entire literature on graphs from rings. The most significant topological property of a graph is its genus. Many researchers have been working on the problem of finding genera of graphs linked with rings such as zero-divisor graphs, total graphs, essential graphs and others. In this regard, one can refer to [15, 17, 18, 20] for details.

In ring theory, ideals play a vital role in the structure of rings. Therefore one common question arose of whether it is possible to define a graph by considering the ideals of a ring R as vertices rather than its elements. As a result of this thinking, Behboodi and Rakeei [7] introduced the notion of annihilating-ideal graphs of rings.

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Over time, a number of graphs have been defined with proper ideals of R as vertices and edges through different ideal theoretic conditions [10, 22, 16]. In 2011, Afkhami and Khashyarmanesh [1] introduced a new graph and called it the cozero-divisor graph of commutative rings. Let $W^*(R)$ be the set of all nonzero nonunits in R, and for $z \in R$, Rz is the ideal generated by z. The cozero-divisor graph $\Gamma'(R)$ of R is the undirected simple graph with $W^*(R)$ as vertex set and such that two distinct vertices x and y in $W^*(R)$ are adjacent if and only if $x \notin Ry$ and $y \notin Rx$. Inspired by their work, Wilkens et al. [21] modified the definition of the cozero-divisor graph by considering nonzero proper principal ideals as vertices and named it the reduced cozero-divisor graph. For a given R, let $\Omega(R)^*$ be the set of all nonzero proper principal ideals of R. The reduced cozero-divisor graph $\Gamma_r(R)$ of R is the simple undirected graph with $\Omega(R)^*$ as vertex set and such that two distinct vertices $\langle a\rangle$ and $\langle b\rangle$ in $\Omega(R)^*$ are adjacent if and only if $\langle a\rangle \nsubseteq \langle b\rangle$ and $\langle b\rangle \nsubseteq \langle a\rangle$. In [21], Amanda Wilkens et al. introduced the reduced cozero-divisor graph $\Gamma_r(R)$ of a ring R (not necessarily commutative). Jesili et al. [12] investigated the toroidal reduced cozero-divisor graph of commutative rings. In this paper, we characterize all Artinian nonlocal rings for which the reduced cozero-divisor is of genus two.

By a graph G we mean a simple finite undirected graph. For any nonempty subset H of vertices of G, the *induced subgraph* generated by H, denoted by $\langle H\rangle$, is the subgraph of G whose vertex set is H and whose edge set is the set of all edges of G that has both ends in H. A graph G is said to be *complete* if every pair of distinct vertices in G are adjacent. A graph G is called *complete bipartite* if the vertex set $V(G)$ can be partitioned into two nonempty disjoint subsets A and B such that every edge in G has one end in A and the other end in B. K_n and $K_{m,n}$ denote the complete graph on n vertices and complete bipartite graph with $|A| = m$ and $|B| = n$, respectively. The *neighborhood* of a vertex v in G is the set of vertices in G which are adjacent with v, and it is denoted by $N_G(v)$ or $N(v)$. The *girth* of G is the length of a shortest cycle in G and is denoted by $\mathrm{gr}(G)$. If G has no cycles, we assume $\mathrm{gr}(G)$ to be infinite. A graph G is said to be *planar* if it can be drawn in the plane so that its edges intersect only at vertices of G. A graph is said to be an *outerplanar graph* if it can be drawn in the plane without crossings such that all vertices are in the unbounded face of that embedding in the plane.

For any non-negative integer n, let $\mathbb{S}_n$ be the orientable surface with n handles. The *genus* of the graph G, denoted by $g(G)$, is the smallest n such that G embeds into $\mathbb{S}_n$. A *subdivision* of G is a graph obtained from G by replacing edges with pairwise internally-disjoint paths. An *n-book embedding* consists of a set of n-half planes called *pages* whose boundaries are bound together on a single line called *spine.* If one can embed the vertices of a graph in the spine of a book, and then place edges in k-pages so that every edge lies in exactly one page, and no two edges cross in a given page, then the embedding is called a k-book embedding. The *book thickness* of a graph G is the smallest integer n for which G has n-book embedding. For details on the notion of embedding of graphs in a surface and book embedding, one can see [19, 8]. For details on graph theory, we refer to [9]. For a basic definition on rings, one may refer to [5].

Now, we present results which will be used in our proofs of this paper. The following is a famous characterization for planar graphs.

Theorem 1.1 (Kuratowski's Theorem [9, p. 153]). *A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

We have the following characterization for outerplanar graphs.

Theorem 1.2 ([11, Proposition 7.3.1]). *A graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.*

Theorem 1.3 ([19, Euler Formula]). *If G is a finite connected graph with n vertices, m edges and genus g, then $n-m+f=2-2g$, where f is the number of faces created when G is minimally embedded on a surface of genus g.*

Lemma 1.4 ([19, Theorem 6.37]). *If $k,\ell \geq 2$ are integers, then*
$$g(K_{k,\ell})=\left\lceil \tfrac{(k-2)(\ell-2)}{4}\right\rceil.$$

Theorem 1.5 ([8, Theorem 2.5]). *Let G be a connected graph. Then the following are true:*

(a) *the book thickness of G is zero if and only if G is a path;*
(b) *the book thickness of G is less than or equal to 1 if and only if G is outerplanar.*

Lemma 1.6 ([4, Lemma 2.1]). *If G is a graph with n vertices, m edges, girth $\mathrm{gr}(G)$ and genus g, then $\frac{m(\mathrm{gr}(G)-2)}{2\,\mathrm{gr}(G)}-\frac{n}{2}+1\leq g(G)$.*

Lemma 1.7 ([19, Corollary 6.15]). *Suppose a simple graph G is connected with $n\geq 3$ vertices, m edges and genus g. If G has no triangles, then $g(G)\geq \left\lceil \frac{m}{4}-\frac{n}{2}+1\right\rceil$.*

Theorem 1.8 ([14, Proposition 4.4.4]). *Let G be a connected graph with $n\geq 3$ vertices, m edges and genus g. Then $g(G)\geq \left\lceil \frac{m}{6}-\frac{n}{2}+1\right\rceil$.*

Theorem 1.9 ([13, Theorem 3.1]). *Let $n\geq 2$ be an integer, F_j be a field for $1\leq j\leq n$, and let $R=F_1\times\cdots\times F_n$. Then $\Gamma_r(R)$ is planar if and only if R is isomorphic to either $F_1\times F_2\times F_3$ or $F_1\times F_2$.*

Theorem 1.10 ([13, Theorem 3.2]). *Let $n\geq 2$ be an integer, $(R_i,\mathfrak{m}_i)$ be a local ring with unique maximal ideal $\mathfrak{m}_i\neq 0$ for $1\leq i\leq n$, and let $R=R_1\times\cdots\times R_n$. Then $\Gamma_r(R)$ is planar if and only if R is isomorphic to $R_1\times R_2$ such that $\mathfrak{m}_i$ is the only nonzero principal ideal in R_i for $1\leq i\leq 2$.*

Theorem 1.11 ([12, Theorem 6]). *Let $(R_i,\mathfrak{m}_i)$ be a local ring with unique maximal ideal $\mathfrak{m}_i\neq\{0\}$ for $1\leq i\leq n$ and let F_j be a field for $1\leq j\leq m$ and $m,n\geq 1$. Let $R=R_1\times\cdots\times R_n\times F_1\times\cdots\times F_m$. Then $\Gamma_r(R)$ is planar if and only if R satisfies the following conditions:*

(1) *$n=m=1$;*
(2) *$\mathfrak{m}_1=\langle a_1\rangle$ is a principal ideal with nilpotency index $k\leq 4$ in general. In particular,*

(i) *if $k = 2$, then $\mathfrak{m}_1 = \langle a_1 \rangle$ is the only nonzero proper principal ideal in R_1;*
(ii) *if $k = 3$, then $\mathfrak{m}_1$ and $\mathfrak{m}_1^2$ are the nonzero principal ideals in R_1;*
(iii) *if $k = 4$, then $\mathfrak{m}_1$, $\mathfrak{m}_1^2$ and $\mathfrak{m}_1^3$ are the nonzero principal ideals in R_1.*

Theorem 1.12 ([12, Theorem 8]). *Let $n \geq 2$ be an integer, $(R_i, \mathfrak{m}_i)$ be a local ring with unique maximal ideal $\mathfrak{m}_i \neq 0$ for $1 \leq i \leq n$ and let $R = R_1 \times \cdots \times R_n$. Let η_i be the nilpotent index of $\mathfrak{m}_i$ for $1 \leq i \leq n$. Then $g(\Gamma_r(R)) = 1$ if and only if R satisfies the following conditions:*

(1) *$n = 2$;*
(2) *$\mathfrak{m}_1 = \langle a_1 \rangle$ and $\mathfrak{m}_2 = \langle b_1 \rangle$ for some $a_1 \in R_1$, $b_1 \in R_2$ and $2 \leq \eta_1, \eta_2 \leq 3$;*
   (i) *if $\eta_1 = 3$ and $\eta_2 = 2$, then $\mathfrak{m}_1$ and $\mathfrak{m}_1^2$ are the only non-trivial principal ideals in R_1, and $\mathfrak{m}_2$ is the only non-trivial principal ideal in R_2;*
   (ii) *if $\eta_1 = 2$ and $\eta_2 = 3$, then $\mathfrak{m}_1$ is the only non-trivial principal ideal in R_1, and $\mathfrak{m}_2$ and $\mathfrak{m}_2^2$ are the only non-trivial principal ideals in R_2.*

Theorem 1.13 ([12, Theorem 9]). *Let $m, n \geq 1$ be integers. Let $(R_i, \mathfrak{m}_i)$ be a local ring with unique maximal ideal $\mathfrak{m}_i \neq \{0\}$ for $1 \leq i \leq n$, and let F_j be a field for $1 \leq j \leq m$. Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$. Then $g(\Gamma_r(R)) = 1$ if and only if R satisfies one of the following conditions:*

(1) *$R \cong R_1 \times F_1 \times F_2$ and $\mathfrak{m}_1$ is the only non-trivial principal ideal in R_1;*
(2) *$R \cong R_1 \times F_1$ and*
   (i) *if $\mathfrak{m}_1 = \langle b_1, b_2 \rangle$, then $\langle b_1 \rangle$, $\langle b_2 \rangle$, $\langle b_1 b_2 \rangle$ and $\langle b_1 + b_2 \rangle$ are the only non-trivial principal ideals of R_1;*
   (ii) *$\mathfrak{m}_1 = \langle b_1 \rangle$ is a principal ideal in R_1 with nilpotency $\eta = 5$ or 6;*
      (a) *if $\eta = 5$, then $\mathfrak{m}$, $\mathfrak{m}^2$, $\mathfrak{m}^3$ and $\mathfrak{m}^4$ are the only non-trivial principal ideals of R_1;*
      (b) *if $\eta = 6$, then $\mathfrak{m}$, $\mathfrak{m}^2$, $\mathfrak{m}^3$, $\mathfrak{m}^4$ and $\mathfrak{m}^5$ are the only non-trivial principal ideals of R_1.*

2. Outerplanarity and genus two characterizations

By following results proved in [12, 13], in this section we aim to characterize all Artinian nonlocal rings whose reduced cozero-divisor graph is outerplanar. Also we characterize all Artinian nonlocal rings whose reduced cozero-divisor graph is of genus two. After these characterizations, we attempt characterizations for the class of Artinian rings. In this regard, we make use of the structure theorem for Artinian rings [5, Theorem 8.7]. An Artinian ring R is isomorphic to the product $R \cong R_1 \times R_2 \times \cdots \times R_n$ of local Artinian rings $(R_i, \mathfrak{m}_i)$.

First, we determine all rings whose reduced cozero-divisor graph is outerplanar. Now we observe the following.

Remark 2.1. Let R be a reduced Artinian nonlocal ring. Here $R \cong R_1 \times \cdots \times R_k$ for some $k \geq 2$ and each $(R_i, \mathfrak{m}_i)$ is a local ring for $1 \leq i \leq k$. If $\mathfrak{m}_i \neq 0$ for some i, then R shall contain a nonzero nilpotent element, which is a contradiction to R being reduced. Hence $\mathfrak{m}_i = 0$ for every i and thus every reduced Artinian nonlocal ring is a direct product of fields.

Theorem 2.2. *Let R be a nonlocal finite ring. Then $\Gamma_r(R)$ is outerplanar if and only if R is isomorphic to either $F_1 \times F_2$ or $R_1 \times F_1$, where F_1 and F_2 are fields and R_1 is a local ring with nonzero maximal ideal $\mathfrak{m}_1$ which is also a principal ideal with nilpotency at most 3.*

Proof. Since R is a nonlocal finite ring, $R \cong R_1 \times \cdots \times R_k$ for some $k \geq 2$ and each $R_i (1 \leq i \leq k)$ is a local ring. Assume that $\Gamma_r(R)$ is outerplanar. Since every outerplanar graph is planar, $\Gamma_r(R)$ is planar.

Case 1. R is reduced.

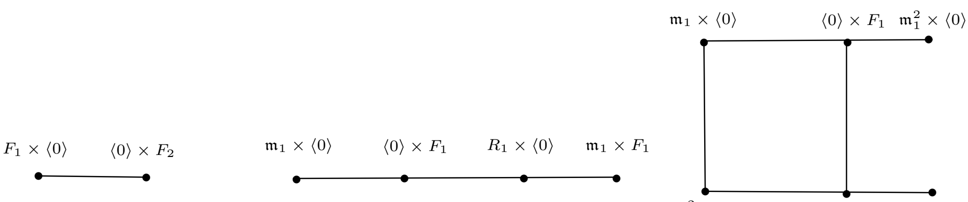
By Remark 2.1, each R_i is a field. By Theorem 1.9, we have either $R = F_1 \times F_2 \times F_3$ or $R = F_1 \times F_2$. Consider the case $R = F_1 \times F_2 \times F_3$. One can easily find $\Gamma_r(F_1 \times F_2 \times F_3)$ contains a subdivision of $K_{2,3}$ as a subgraph corresponding to vertex partitions $\{\langle 0\rangle \times F_2 \times \langle 0\rangle, \langle 0\rangle \times F_2 \times F_3\}$, $\{F_1 \times \langle 0\rangle \times \langle 0\rangle, F_1 \times \langle 0\rangle \times F_3, \langle 0\rangle \times \langle 0\rangle \times F_3\}$ and a subdivision of the edge joining $\langle 0\rangle \times F_2 \times F_3$ and $\langle 0\rangle \times \langle 0\rangle \times F_3$ through the vertex $F_1 \times F_2 \times \langle 0\rangle$. By Theorem 1.2, $\Gamma_r(F_1 \times F_2 \times F_3)$ is not outerplanar, which is a contradiction. Hence $R = F_1 \times F_2$, where each F_i is a field.

Case 2. R is non-reduced.

By Theorems 1.10 and 1.11, one needs to check only for the rings: $R_1 \times R_2$ and $R_1 \times F_1$, where each R_i is a local ring with non-zero maximal principal ideal $\mathfrak{m}_i$ and F_1 is a field. Note that $\Gamma_r(R_1 \times R_2)$ contains a subdivision of $K_{2,3}$ as a subgraph corresponding to vertex partitions $\{\langle 0\rangle \times R_2, R_1 \times \langle 0\rangle\}$, $\{\mathfrak{m}_1 \times \langle 0\rangle, \mathfrak{m}_1 \times \mathfrak{m}_2, R_1 \times \mathfrak{m}_2\}$ and a subdivision of edges joining $R_1 \times \langle 0\rangle$ and $\mathfrak{m}_1 \times \langle 0\rangle$, $R_1 \times \langle 0\rangle$ and $R_1 \times \mathfrak{m}_2$ through the vertices $\langle 0\rangle \times \mathfrak{m}_2$, $\mathfrak{m}_1 \times R_2$, respectively. Here we arrive at a contradiction by Theorem 1.2.

Consider the ring $R = R_1 \times F_1$. Let η_1 be the nilpotent index $\mathfrak{m}_1$ in R_1. Suppose that $\eta_1 = 4$. Then $\Gamma_r(R)$ contains $K_{2,3}$ as a subgraph with vertex partitions $\{\langle 0\rangle \times F_1, \mathfrak{m}_1^3 \times F_1\}$ and $\{\mathfrak{m}_1^2 \times \langle 0\rangle, R_1 \times \langle 0\rangle, \mathfrak{m}_1 \times \langle 0\rangle\}$, a contradiction. Thus the nilpotent index $\eta_1 \leq 3$.

The converse follows from Figure 2.1. $\square$



(a) $\Gamma_r(F_1 \times F_2)$

(b) $\Gamma_r(R_1 \times F_1)$ and $\eta_1 = 2$

(c) $\Gamma_r(R_1 \times F_1)$ and $\eta_1 = 3$

FIGURE 2.1.

Now, let us find out the classes of reduced rings whose reduced cozero-divisor graphs can be embedded in $\mathbb{S}_2$.

Lemma 2.3. *Let R be a reduced Artinian ring with at least four maximal ideals. Then $g(\Gamma_r(R)) \geq 3$.*

Proof. Since R is a reduced Artinian ring, R is a direct product of fields. Since R contains at least four maximal ideals, $R = F_1 \times \cdots \times F_n$ and $n \geq 4$. Consider the set $A = \{J_1 = F_1 \times \langle 0\rangle \times \langle 0\rangle \times \langle 0\rangle \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_2 = \langle 0\rangle \times F_2 \times \langle 0\rangle \times \langle 0\rangle \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_3 = \langle 0\rangle \times \langle 0\rangle \times F_3 \times \langle 0\rangle \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_4 = \langle 0\rangle \times \langle 0\rangle \times \langle 0\rangle \times F_4 \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_5 = F_1 \times F_2 \times \langle 0\rangle \times \langle 0\rangle \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_6 = F_1 \times \langle 0\rangle \times F_3 \times \langle 0\rangle \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_7 = F_1 \times \langle 0\rangle \times \langle 0\rangle \times F_4 \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_8 = \langle 0\rangle \times F_2 \times F_3 \times \langle 0\rangle \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_9 = \langle 0\rangle \times F_2 \times \langle 0\rangle \times F_4 \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_{10} = \langle 0\rangle \times \langle 0\rangle \times F_3 \times F_4 \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_{11} = F_1 \times F_2 \times F_3 \times \langle 0\rangle \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_{12} = F_1 \times \langle 0\rangle \times F_3 \times F_4 \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_{13} = F_1 \times F_2 \times \langle 0\rangle \times F_4 \times \langle 0\rangle \times \cdots \times \langle 0\rangle, J_{14} = \langle 0\rangle \times F_2 \times F_3 \times F_4 \times \langle 0\rangle \times \cdots \times \langle 0\rangle\}$ of $\Gamma_r(R)$ and the induced subgraph $\langle A\rangle$.

Note that $N_{\langle A\rangle}(J_1) = \{J_2, J_3, J_4, J_8, J_9, J_{10}, J_{14}\}$, $N_{\langle A\rangle}(J_2) = \{J_1, J_3, J_4, J_6, J_7, J_{10}, J_{12}\}$, $N_{\langle A\rangle}(J_3) = \{J_1, J_2, J_4, J_5, J_7, J_9, J_{13}\}$, $N_{\langle A\rangle}(J_4) = \{J_1, J_2, J_3, J_5, J_6, J_8, J_{11}\}$, $N_{\langle A\rangle}(J_5) = \{J_3, J_4, J_6, J_7, J_8, J_9, J_{10}, J_{12}, J_{14}\}$, $N_{\langle A\rangle}(J_6) = \{J_2, J_4, J_5, J_7, J_8, J_9, J_{10}, J_{13}, J_{14}\}$, $N_{\langle A\rangle}(J_7) = \{J_2, J_3, J_5, J_6, J_8, J_9, J_{10}, J_{11}, J_{14}\}$, $N_{\langle A\rangle}(J_8) = \{J_1, J_4, J_5, J_6, J_7, J_9, J_{10}, J_{12}, J_{13}\}$, $N_{\langle A\rangle}(J_9) = \{J_1, J_3, J_5, J_6, J_7, J_8, J_{10}, J_{11}, J_{12}\}$, $N_{\langle A\rangle}(J_{10}) = \{J_1, J_2, J_5, J_6, J_7, J_8, J_9, J_{11}, J_{13}\}$, $N_{\langle A\rangle}(J_{11}) = \{J_4, J_7, J_9, J_{10}, J_{12}, J_{13}, J_{14}\}$, $N_{\langle A\rangle}(J_{12}) = \{J_2, J_5, J_8, J_9, J_{11}, J_{13}, J_{14}\}$, $N_{\langle A\rangle}(J_{13}) = \{J_3, J_6, J_8, J_{10}, J_{11}, J_{12}, J_{14}\}$, $N_{\langle A\rangle}(J_{14}) = \{J_1, J_5, J_6, J_7, J_{11}, J_{12}, J_{13}\}$. Thus we have an induced subgraph $\langle A\rangle$ of $\Gamma_r(R)$ with $n = 14$ vertices and $m = 55$ edges. By Theorem 1.8, we get $g(\Gamma_r(R)) \geq 3$. □

Clearly, $\Gamma_r(F_1 \times F_2 \times F_3 \times F_4)$ is a subgraph of $\Gamma_r(R_1 \times R_2 \times R_3 \times R_4)$, where each R_i is a local ring and each F_j is a field. By Lemma 2.3, $g(\Gamma_r(R_1 \times R_2 \times R_3 \times R_4)) \geq 3$. Therefore, to characterize the genus two reduced cozero-divisor graphs, it is enough to look into the cases $R_1 \times R_2 \times R_3$ and $R_1 \times R_2$. By Theorem 1.9, we cannot take all R_i's to be fields, so we consider the non-reduced case only.

Theorem 2.4. *Let $(R_1, \mathfrak{m}_1)$ be a local ring with $\mathfrak{m}_1 \neq 0$ and η_1 be the nilpotent index of $\mathfrak{m}_1$. Let $R = R_1 \times F_1 \times F_2$, where $F_1$and F_2 are fields. Then the following are true:*

(1) *if $\mathfrak{m}_1 = \langle a_1\rangle$ is a principal ideal with nilpotency $\eta_1 \geq 3$, then $g(\Gamma_r(R)) \geq 3$;*
(2) *if $\mathfrak{m}_1 = \langle a_1, a_2, \ldots, a_\ell\rangle$, $a_i \in R_1$ and $\ell \geq 2$, then $g(\Gamma_r(R)) \geq 3$.*

Proof. Let $R = R_1 \times F_1 \times F_2$. To prove (1), assume that $\mathfrak{m}_1$ is principal with nilpotency $\eta_1 \geq 3$. Consider the subgraph induced by $\{I_1 = \mathfrak{m}_1 \times \langle 0\rangle \times R_3, I_2 = \mathfrak{m}_1^2 \times \langle 0\rangle \times R_3, I_3 = R_1 \times \langle 0\rangle \times R_3, I_4 = \langle 0\rangle \times \langle 0\rangle \times R_3, I_5 = R_1 \times \langle 0\rangle \times \langle 0\rangle, I_6 = \langle 0\rangle \times R_2 \times \langle 0\rangle, I_7 = \mathfrak{m}_1 \times R_2 \times \langle 0\rangle, I_8 = \mathfrak{m}_1^2 \times R_2 \times \langle 0\rangle, I_9 = \langle 0\rangle \times R_2 \times R_3, I_{10} = R_1 \times R_2 \times \langle 0\rangle, I_{11} = \mathfrak{m}_1 \times \langle 0\rangle \times \langle 0\rangle, I_{12} = \mathfrak{m}_1^2 \times R_1 \times R_3\}$. It contains a subdivision of $K_{5,5}$ with vertex partitions $\{I_1, I_2, I_3, I_4, I_5\}$, $\{I_6, I_7, I_8, I_9, I_{10}\}$ and the edges joining I_4 and I_9, I_5 and I_{10} through the vertices I_{11} and I_{12}, respectively. By using the Lemma 1.4, we get that $g(\Gamma_r(R)) \geq 3$.

To prove (2), we assume that $\mathfrak{m}_1 = \langle a_1, a_2, \ldots, a_\ell\rangle$, $a_i \in R_1$ and $\ell \geq 2$. Then the subgraph induced by $\{H_1 = \langle a_2\rangle \times \langle 0\rangle \times \langle 0\rangle, H_2 = \langle a_2\rangle \times R_2 \times \langle 0\rangle, H_3 = \langle a_2\rangle \times \langle 0\rangle \times R_3, H_4 = R_1 \times \langle 0\rangle \times \langle 0\rangle, H_5 = R_1 \times R_2 \times \langle 0\rangle, H_6 = \langle 0\rangle \times R_2 \times R_3, H_7 =$

$\langle a_1\rangle\times\langle 0\rangle\times R_3, H_8 = \langle a_1\rangle\times R_2\times R_3, H_9 = \langle 0\rangle\times\langle 0\rangle\times R_3, H_{10} = \langle a_1\rangle\times R_2\times\langle 0\rangle, H_{11} = \langle 0\rangle\times R_2\times\langle 0\rangle, H_{12} = \langle a_2\rangle\times R_2\times R_3\}$ contains a subdivision of $K_{5,5}$ with vertex partitions $\{H_1, H_2, H_3, H_4, H_5\}$, $\{H_6, H_7, H_8, H_9, H_{10}\}$ and a subdivision of edges joining H_3 and H_9, H_5 and H_{10} through the vertices H_{11} and H_{12}, respectively. By Lemma 1.4, we get that $g(\Gamma_r(R)) \geq 3$. □

Theorem 2.5. *Let $(R_i, \mathfrak{m}_i)$ be a local ring for $1 \leq i \leq 3$, and let $R = R_1\times R_2\times R_3$. If $\mathfrak{m}_i = 0$ for at most one $i \in \{1, 2, 3\}$, then $g(\Gamma_r(R)) \geq 3$.*

Proof. Let $\mathfrak{m}_i = 0$ for at most one i. Without loss of generality, assume that $\mathfrak{m}_3 = 0$ and hence R_3 is a field. Let $Y_1 = \langle 0\rangle\times U_2\times\langle 0\rangle$, $Y_2 = \langle 0\rangle\times R_2\times\langle 0\rangle$, $Y_3 = \langle 0\rangle\times U_2\times R_3$, $Y_4 = \langle 0\rangle\times R_2\times R_3$, $Y_5 = U_1\times U_2\times\langle 0\rangle$, $Y_6 = U_1\times R_2\times\langle 0\rangle$, $Y_7 = R_1\times U_2\times\langle 0\rangle$, $Y_8 = R_1\times\langle 0\rangle\times\langle 0\rangle$, $Y_9 = U_1\times\langle 0\rangle\times R_3$, $Y_{10} = \langle 0\rangle\times\langle 0\rangle\times R_3$, $Y_{11} = R_1\times\langle 0\rangle\times R_3$, $Y_{12} = R_1\times R_2\times\langle 0\rangle$, $Y_{13} = U_1\times U_2\times R_3$, where U_1 and U_2 are nonzero proper principal ideals in R_1 and R_2, respectively. Let $B = \{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}, Y_{13}\}$. Then the subgraph $\langle B\rangle$ contains a subdivision of $K_{7,4}$ with vertex partitions $\{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7\}$, $\{Y_8, Y_9, Y_{10}, Y_{11}\}$ and the edges joining Y_3 and Y_{10}, Y_7 and Y_8 through the vertices Y_{12} and Y_{13}, respectively. By applying Lemma 1.4, we observe that $g(\langle B\rangle) \geq 3$. Since $\langle B\rangle$ is the subgraph of $\Gamma_r(R)$, $g(\Gamma_r(R)) \geq 3$. □

Now we end this section with the following main theorem.

Theorem 2.6. *Let $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ be two local rings, and let $R = R_1\times R_2$. Let η_i be the nilpotent index of $\mathfrak{m}_i$ for $i = 1, 2$. Then $g(\Gamma_r(R)) = 2$ if and only if any of the following are true:*

(1) *R_2 is a field and $\mathfrak{m}_1$ is a principal ideal in R_1 with nilpotency $\eta_1 \leq 7$;*
(2) *$\mathfrak{m}_1 = \langle x_1\rangle$ is a principal ideal in R_1 and $\mathfrak{m}_2 = \langle y_1\rangle$ is a principal ideal in R_2;*
   (a) *if $\eta_1 = 2$ and $\eta_2 = 4$, then $\mathfrak{m}_1$ is the only non-trivial principal ideal in R_1, and $\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_2^3$ are the only non-trivial principal ideals in R_2;*
   (b) *if $\eta_1 = 4$ and $\eta_2 = 2$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ are the only non-trivial principal ideals in R_1, and $\mathfrak{m}_2$ is the only non-trivial principal ideal in R_2.*

Proof. Assume that $g(\Gamma_r(R)) = 2$. Suppose that $\mathfrak{m}_i = 0$ for all i. Then both R_1 and R_2 are fields, and hence $g(\Gamma_r(R))$ is planar as proved in Theorem 1.9. This is a contradiction to the assumption that $g(\Gamma_r(R)) = 2$. So, now we look into the cases where either $\mathfrak{m}_1 = 0$ or $\mathfrak{m}_2 = 0$ but not both.

Case 1. Assume that $\mathfrak{m}_1 \neq 0$ and $\mathfrak{m}_2 = 0$.

Here R_2 is a field. Since R_1 is Artinian, every ideal in R_1 is finitely generated. Let $\varphi = \{c_1, c_2, \ldots, c_s : c_j \in R_1 \text{ for } 1 \leq j \leq s\}$ be a minimal generating set for $\mathfrak{m}_1$ in R_1. Then $\langle c_r\rangle \nsubseteq \langle c_t\rangle$ for all $r \neq t, 1 \leq r, t \leq s$. Suppose $s \geq 3$. Let $W_1 = R_1\times\langle 0\rangle, W_2 = \langle c_1 + c_2 + c_3\rangle\times\langle 0\rangle, W_3 = \langle c_2 + c_3\rangle\times\langle 0\rangle, W_4 = \langle c_1 + c_3\rangle\times\langle 0\rangle, W_5 = \langle c_1 + c_2\rangle\times\langle 0\rangle, W_6 = \langle c_3\rangle\times\langle 0\rangle, W_7 = \langle c_2\rangle\times\langle 0\rangle, W_8 = \langle 0\rangle\times F_1, W_9 = \langle c_1\rangle\times F_1, W_{10} = \langle c_2\rangle\times F_1, W_{11} = \langle c_3\rangle\times F_1, W_{12} = \langle c_1+c_2\rangle\times F_1, W_{13} = \langle c_1\rangle\times(0)$ and $A = \{W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12}, W_{13}\} \subset \Omega(R)^*$. Then the subgraph induced by A contains a subdivision of $K_{7,4}$ with partition subsets

$\{W_1, W_2, W_3, W_4, W_5, W_6, W_7\}$ and $\{W_8, W_9, W_{10}, W_{11}\}$ and a subdivision of edges joining the vertices W_6 and W_{11}, W_7 and W_{10} through W_{12} and W_{13}, respectively. Using Theorem 1.4, we get that $g(\Gamma_r(R)) \geq 3$, which is a contradiction. This gives that either $s = 2$ or $s = 1$.

Suppose that $s = 2$. If $\mathfrak{m}_1^2 = 0$, by Theorem 1.11, $\Gamma_r(R)$ is planar, which is a contradiction to $g(\Gamma_r(R)) = 2$. Thus we have $\mathfrak{m}_1 = \langle c_1, c_2\rangle$ and $\mathfrak{m}_1^2 \neq 0$. Suppose $c_i^2 \neq 0$ for some i. Without loss of generality, let us assume that $c_1^2 \neq 0$. Consider the principal ideals $I_1 = \langle 0\rangle \times F_1$, $I_2 = \langle c_1c_2\rangle \times F_1$, $I_3 = \left\langle c_1^2\right\rangle \times F_1$, $I_4 = \langle c_1\rangle \times F_1$, $I_5 = \langle c_2\rangle \times F_1$, $I_6 = \langle c_1 + c_2\rangle \times \langle 0\rangle$, $I_7 = \left\langle c_1^2 + c_2\right\rangle \times \langle 0\rangle$, $I_8 = R_1 \times \langle 0\rangle$, $I_9 = \langle c_1\rangle \times \langle 0\rangle$, $I_{10} = \langle c_2\rangle \times \langle 0\rangle$, $I_{11} = \left\langle c_1^2 + c_2\right\rangle \times F_1$, $I_{12} = \left\langle c_1^2\right\rangle \times \langle 0\rangle$. Then the subgraph induced by $B = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}, I_{11}, I_{12}\} \subset \Omega(R)^*$ contains a subdivision of $K_{5,5}$ with vertex partitions $\{I_1, I_2, I_3, I_4, I_5\}$, $\{I_6, I_7, I_8, I_9, I_{10}\}$ and subdivision of edges joining the vertices I_4 and I_9, I_5 and I_{10} through I_{11} and I_{12}, respectively. By Theorem 1.4, we get that $g(\Gamma_r(R)) = 3$, which is a contradiction. Hence, $c_i^2 = 0$ for $i = 1, 2$, and so $\mathfrak{m}_1^2 = \langle c_1c_2\rangle$. Note that $\mathfrak{m}_1^3$ is generated by $c_1^2c_2$, $c_1c_2^2$, c_1^3 and c_2^3. Since $c_i^2 = 0$ for $i = 1, 2$, we get that $\mathfrak{m}_1^3 = 0$. This implies that $\langle c_1\rangle$, $\langle c_2\rangle$, $\langle c_1c_2\rangle$ and $\langle c_1 + c_2\rangle$ are the only non-trivial principal ideals in R_1. By Theorem 1.13 (2)(i), we get that $g(\Gamma_r(R)) = 1$, which is a contradiction. Hence $s = 1$ and $\mathfrak{m}_1 = \langle c_1\rangle$ is the principal ideal in R_1.

Since R_1 is Artinian, $\mathfrak{m}_1^{\eta_1} = (0)$, $\mathfrak{m}_1^{\eta_1 - 1} \neq (0)$ for some $\eta_1 \geq 2$. Now we claim that $\eta_1 \leq 7$. Suppose that $\eta_1 \geq 8$. Let H be the subgraph of $\Gamma_r(R)$ induced by the vertex set $\{\langle 0\rangle \times F_1, \mathfrak{m}_1^7 \times F_1, \mathfrak{m}_1^6 \times F_1, \mathfrak{m}_1^5 \times F_1, \mathfrak{m}_1^4 \times F_1, \mathfrak{m}_1^3 \times F_1, \mathfrak{m}_1 \times \langle 0\rangle, \mathfrak{m}_1^2 \times \langle 0\rangle, R_1 \times \langle 0\rangle, \mathfrak{m}_1^3 \times \langle 0\rangle, \mathfrak{m}_1^4 \times \langle 0\rangle, \mathfrak{m}_1^5 \times \langle 0\rangle\}$. Then the graph H is as shown in Figure 2.2 with $v = 12$ vertices and $e = 30$ edges.

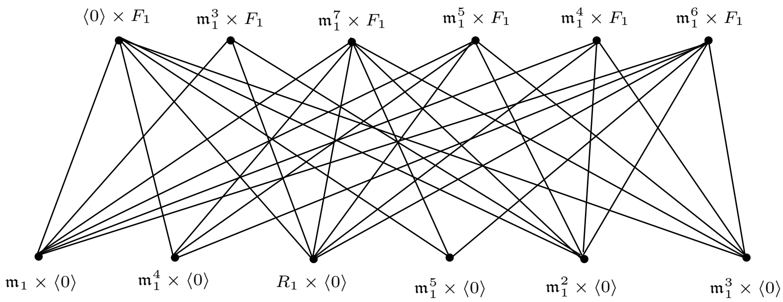


FIGURE 2.2. The graph H

Clearly, the graph H has no triangles. By Theorem 1.7 on the graph H, we get $g(H) \geq 3$, which in turn gives that $g(\Gamma_r(R)) \geq 3$, which is a contradiction to the assumption. Therefore $\eta_1 \leq 7$.

Case 2. Assume that $\mathfrak{m}_1 \neq 0$ and $\mathfrak{m}_2 \neq 0$.

In this case both R_1 and R_2 are not fields. Since R_i is Artinian, every ideal in R_i is finitely generated. Let $\varphi_1 = \{x_1, x_2, \ldots, x_t : x_j \in R_1 \text{ for } 1 \leq j \leq t\}$ and $\varphi_2 = \{y_1, y_2, \ldots, y_k : y_i \in R_2 \text{ for } 1 \leq i \leq k\}$ be minimal generating sets of $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively. Then $\langle x_i \rangle \nsubseteq \langle x_j \rangle$ for all $i \neq j$ and $\langle y_i \rangle \nsubseteq \langle y_j \rangle$ for all $i \neq j$.

Suppose $t \geq 2$ and $k \geq 2$. Let $B = \{\langle x_1 \rangle \times \langle 0 \rangle, \langle x_1 \rangle \times \langle y_1 \rangle, \langle x_2 \rangle \times \langle y_1 \rangle, \langle x_1 + x_2 \rangle \times \langle 0 \rangle, \langle x_1 + x_2 \rangle \times \langle y_1 \rangle, R_1 \times \langle 0 \rangle, R_1 \times \langle y_1 \rangle, \langle 0 \rangle \times \langle y_2 \rangle, \langle 0 \rangle \times \langle y_1 + y_2 \rangle, \langle 0 \rangle \times R_2, \langle x_2 \rangle \times \langle y_2 \rangle\} \subset \Omega(R)^*$. Then the subgraph induced by B contains $K_{7,4}$ as a subgraph with vertex partitions $\{\langle x_1 \rangle \times \langle 0 \rangle, \langle x_1 \rangle \times \langle y_1 \rangle, \langle x_2 \rangle \times \langle y_1 \rangle, \langle x_1 + x_2 \rangle \times \langle 0 \rangle, \langle x_1 + x_2 \rangle \times \langle y_1 \rangle, R_1 \times \langle 0 \rangle, R_1 \times \langle y_1 \rangle\}$ and $\{\langle 0 \rangle \times \langle y_2 \rangle, \langle 0 \rangle \times \langle y_1 + y_2 \rangle, \langle 0 \rangle \times R_2, \langle x_2 \rangle \times \langle y_2 \rangle\}$. By Lemma 1.4, we get that $g(\Gamma_r(R)) \geq 3$, which is a contradiction. This gives that $t = 1$ or $k = 1$. Without loss of generality, let us assume that $t = 1$.

Suppose that $k \geq 2$. Let $X = \{J_1 = \langle 0 \rangle \times \langle y_1 + y_2 \rangle$, $J_2 = \langle 0 \rangle \times R_2$, $J_3 = \mathfrak{m}_1 \times \langle y_1 + y_2 \rangle$, $J_4 = \mathfrak{m}_1 \times R_2$, $I_1 = R_1 \times \langle 0 \rangle$, $I_2 = R_1 \times \langle y_1 \rangle$, $I_3 = R_1 \times \langle y_2 \rangle, Q_1 = \langle 0 \rangle \times \langle y_1 \rangle, Q_2 = \langle 0 \rangle \times \langle y_2 \rangle, Q_3 = \mathfrak{m}_1 \times \langle 0 \rangle, Q_4 = \mathfrak{m}_1 \times \langle y_2 \rangle, Q_5 = \mathfrak{m}_1 \times \langle y_1 \rangle\} \subset \Omega(R)^*$. Let $H = \{J_1, J_2, J_3, J_4, I_1, I_2, I_3\}$ and $H' = \langle H \rangle - \{J_2 J_3, I_2 I_3\}$. Since $J_i I_j \in E(\Gamma_r(R))$ for all i, j and $H' \cong K_{4,3}$. Note that the vertex Q_1 is adjacent to Q_2, Q_3, Q_4, and the vertex Q_2 is adjacent to Q_5. This indicates that we must insert the vertices Q_1, Q_2, Q_3, Q_4 and Q_5 in a same face. Also observe that Q_1 is adjacent to $\{I_1, I_3\}$, Q_2 is adjacent to $\{Q_3, I_1, I_2\}$, Q_3 is adjacent to $\{J_1, J_2\}$, Q_4 is adjacent to $\{Q_5, J_1, J_2, I_1, I_2\}$ and Q_5 is adjacent to $\{J_1, J_2, I_1, I_3\}$.

If we try to embed the graph H' in $\mathbb{S}_1$, then the possible length of the faces will be 4, 6 or 8. Since we have to insert the vertices Q_1, Q_2, Q_3, Q_4, Q_5 to obtain the subgraph $\langle X \rangle$ of $\Gamma_r(R)$, it is not possible to consider the embedding of the graph $\langle X \rangle$ in $\mathbb{S}_1$ without edge crossings.

Therefore, we consider the embedding of $K_{4,3}$ in $\mathbb{S}_2$. Also we must have a face of length greater than 8. However, there is no such embedding in $\mathbb{S}_2$. Hence, we cannot insert Q_1, Q_2, Q_3, Q_4 and Q_5 in the same face without edge crossings. Thus, $g(\Gamma_r(R)) \geq 3$, which is a contradiction. Therefore, $k = 1$.

Since each R_j is Artinian, $\mathfrak{m}_j^{\eta_j} = (0)$, $\mathfrak{m}_j^{\eta_j - 1} \neq (0)$ for some $\eta_j \geq 2$. Suppose $\eta_j \geq 3$ for all j. Let $S = \{U_1, U_2, U_3, U_4, V_1, V_2, V_3, V_4, W_1, W_2, Y_1, Y_2, X_1, X_2\} \subset \Omega(R)^*$, where $U_1 = \langle 0 \rangle \times \mathfrak{m}_2$, $U_2 = \langle 0 \rangle \times R_2$, $U_3 = \mathfrak{m}_1^2 \times \mathfrak{m}_2$, $U_4 = \mathfrak{m}_1^2 \times R_2$, $V_1 = \mathfrak{m}_1 \times \langle 0 \rangle$, $V_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $V_3 = R_1 \times \langle 0 \rangle$, $V_4 = R_1 \times \mathfrak{m}_2^2$, $W_1 = \langle 0 \rangle \times \mathfrak{m}_2^2$, $W_2 = \mathfrak{m}_1^2 \times \langle 0 \rangle$, $Y_1 = \mathfrak{m}_1 \times R_2$, $Y_2 = R_1 \times \mathfrak{m}_2$, $X_1 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $X_2 = \mathfrak{m}_1^2 \times \mathfrak{m}_2^2$. Let $S' = \{U_1, U_2, U_3, U_4, V_1, V_2, V_3, V_4\}$ and $G = \langle S' \rangle - \{U_2 U_3, V_2 V_3\}$. One can observe that $U_i V_j \in E(\Gamma_r(R))$ for all i, j, and so G is isomorphic to $K_{4,4}$. Note that the vertex W_1 is adjacent to W_2, V_1, V_3, and the vertex W_2 is adjacent to U_1, U_2. Due to this, we must insert the vertices W_1, W_2 in the same face of length at least 8 to avoid crossings. Similarly, the vertex Y_1 is adjacent to Y_2, V_3, V_4, and the vertex Y_2 is adjacent to U_2, U_4. With this information, we must insert vertices Y_1, Y_2 in the same face of length at least 8. Since we need at least two faces of length at least 8, we take into account the embedding of $K_{4,4}$ in $\mathbb{S}_2$. Also note that the vertex X_1 is adjacent to U_2, U_4, V_3, V_4, and the vertex X_2 is adjacent to U_1, U_2, V_1, V_3. It is clear that we cannot insert either of the sets $\{Y_1, Y_2, X_1\}$ or $\{W_1, W_2, X_2\}$ without edge crossings. This means that no such embedding exists in the embedding of $K_{4,4}$ in $\mathbb{S}_2$, which yields $g(\Gamma_r(S)) \geq 3$. This in turn gives $g(\Gamma_r(R)) \geq 3$, which is a contradiction. Hence $\eta_j = 2$ for some j. Let us take $\eta_1 = 2$.

Suppose that $\eta_2 \geq 5$. Let $G = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11}, X_{12}, X_{13}, X_{14}, X_{15}, X_{16}\} \subset \Omega(R)^*$ and $G' = \langle G \rangle - \{X_1X_8, X_5X_7, X_5X_8, X_9X_{12}, X_9X_{16}, X_{10}X_{12}\}$, where $X_1 = \langle 0 \rangle \times \mathfrak{m}_2$, $X_2 = \langle 0 \rangle \times \mathfrak{m}_2^2$, $X_3 = \langle 0 \rangle \times \mathfrak{m}_2^3$, $X_4 = \langle 0 \rangle \times \mathfrak{m}_2^4$, $X_5 = \langle 0 \rangle \times R_2$, $X_6 = \mathfrak{m}_1 \times \langle 0 \rangle$, $X_7 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $X_8 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $X_9 = \mathfrak{m}_1 \times \mathfrak{m}_2^3$, $X_{10} = \mathfrak{m}_1 \times \mathfrak{m}_2^4$, $X_{11} = \mathfrak{m}_1 \times R_2$, $X_{12} = R_1 \times \langle 0 \rangle$, $X_{13} = R_1 \times \mathfrak{m}_2$, $X_{14} = R_1 \times \mathfrak{m}_2^2$, $X_{15} = R_1 \times \mathfrak{m}_2^3$, $X_{16} = R_1 \times \mathfrak{m}_2^4$. Then the induced subgraph G' of $\Gamma_r(R)$ has $n = 16$ vertices, $m = 39$ edges, and its girth is 4. Using the Lemma 1.8 to the graph G', we obtain $g(G') \geq 3$. Since G' is the subgraph of $\Gamma_r(R)$, we get $g(\Gamma_r(R)) \geq 3$. This contradicts our assumption. Thus, we have $\eta_2 \leq 4$. Because of Theorem 1.10, we conclude that $\eta_2 = 4$. Therefore, $\mathfrak{m}_2$, $\mathfrak{m}_2^2$, $\mathfrak{m}_2^3$, and $\mathfrak{m}_2^4$ are the only nonzero proper principal ideals in R_2. In an analogous manner, we can prove that if $\eta_2 = 2$, then R_1 contains precisely $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$, and $\mathfrak{m}_1^4$ as the only nonzero proper principal ideals.

The converse follows from the embeddings given in Figures 2.3, 2.4 and 2.5. $\square$

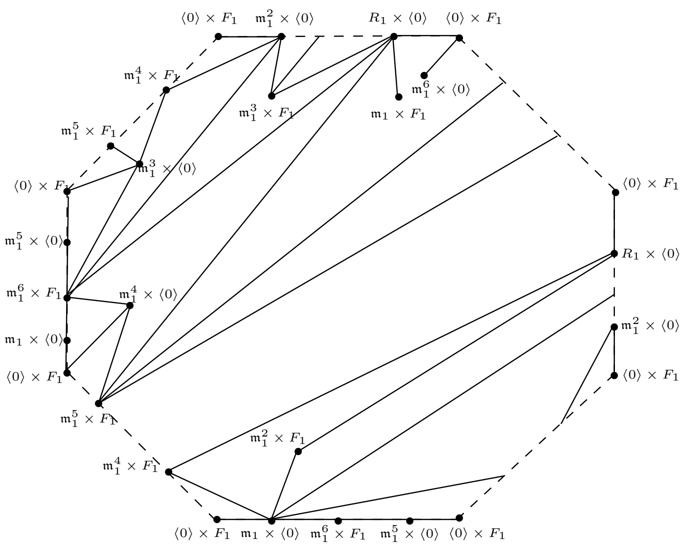


Figure 2.3. Embedding of $\Gamma_r(R_1 \times F_1)$ and $\eta_1 = 7$ on $\mathbb{S}_2$

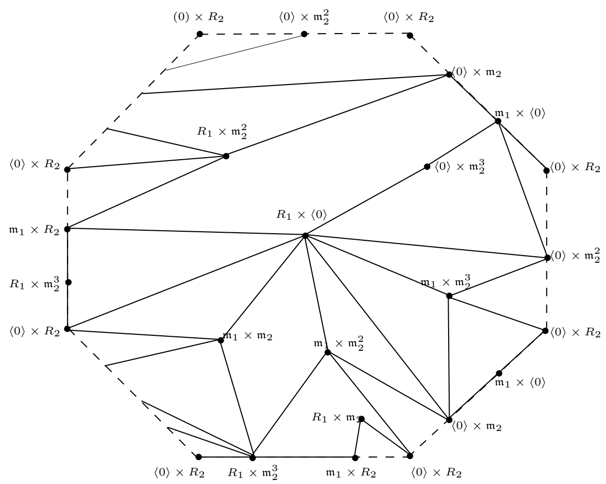


FIGURE 2.4. Embedding of $\Gamma_r(R_1 \times R_2)$ with $\eta_1 = 2$ and $\eta_2 = 4$ on $\mathbb{S}_2$

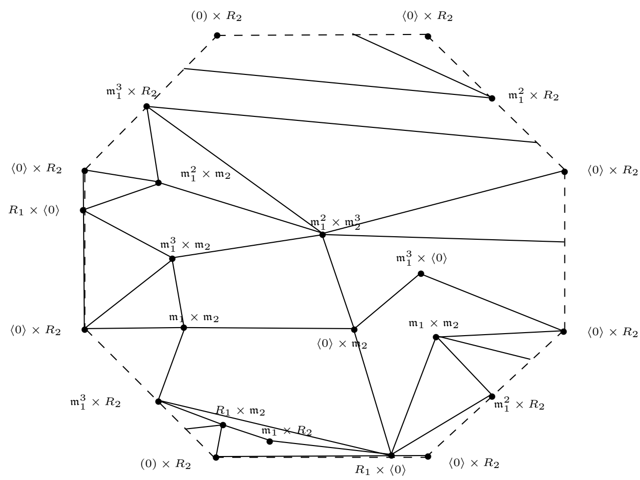


FIGURE 2.5. Embedding of $\Gamma_r(R_1 \times R_2)$ with $\eta_1 = 4$ and $\eta_2 = 2$ on $\mathbb{S}_2$

3. Book thickness of reduced cozero-divisor graph

In this section, we determine the book thickness of the reduced cozero-divisor graph whose genus is at most one. First of all, we find out the book thickness of planar reduced cozero-divisor graphs arising from rings listed in Theorems 1.9, 1.10 and 1.11. In the next theorem, we prove that all planar reduced cozero-divisor graphs have book thickness at most two.

Theorem 3.1. *Let R be a commutative Artinian nonlocal ring with identity whose reduced cozero-divisor graph is planar. Then the following are true:*

(1) *the book thickness of $\Gamma_r(R)$ is 0 if and only if R is isomorphic to either $F_1 \times F_2$ or $R_1 \times F_1$, where F_1 and F_2 are fields and R_1 is a local ring with nonzero maximal principal ideal of nilpotent index 2;*

(2) *the book thickness of $\Gamma_r(R)$ is 1 if and only if R is isomorphic to $R_1 \times F_1$, where F_1 is a field and R_1 is a local ring with nonzero maximal principal ideal of nilpotent index 3;*

(3) *the book thickness of $\Gamma_r(R)$ is 2 if and only if R is isomorphic to one of the following rings:*
   - (a) *$F_1 \times F_2 \times F_3$, where each F_j is a field for $1 \leq j \leq 3$;*
   - (b) *$R_1 \times R_2$, where each R_i is a local ring with $\mathfrak{m}_i \neq 0$ as the only nonzero proper principal ideal in R_i for $i = 1, 2$;*
   - (c) *$R_1 \times F_1$, where F_1 is a field and R_1 is a local ring and $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$ are the only proper principal ideals in R_1.*

Proof. Parts (1) and (2) follow from Figure 2.1 and Theorems 1.5 and 2.2.

To prove (3), we need to consider the remaining commutative Artinian nonlocal rings whose reduced cozero-divisor graph is planar as given in Theorems 1.9, 1.10 and 1.11. Note that all remaining rings are considered in (3) (a), (b) and (c). Since they are not outerplanar, they should have book thickness greater than or equal to 2. However, Figures 3.1, 3.2, and 3.3 give 2-book embeddings for these rings and hence the proof is complete. □

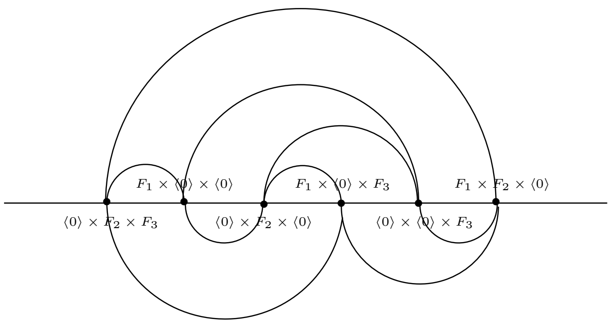


Figure 3.1. Two-page book embedding of $\Gamma_r(F_1 \times F_2 \times F_3)$

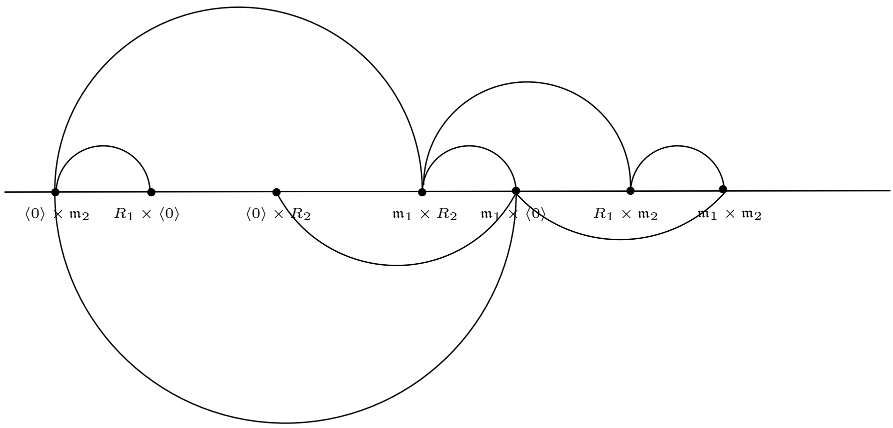


FIGURE 3.2. Two-page embedding of $\Gamma_r(R_1 \times R_2)$

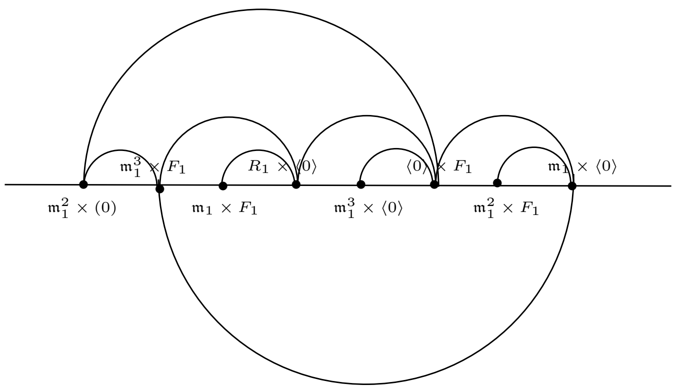


FIGURE 3.3. Two-page embedding of $\Gamma_r(R_1 \times F_1)$ with $\eta_1 = 4$

For the class of toroidal reduced cozero-divisor graphs, the book thickness is obtained in the following theorem.

Theorem 3.2. *Let R be a commutative Artinian nonlocal ring with identity whose reduced cozero-divisor graph is toroidal.*

(1) *The book thickness of $\Gamma_r(R)$ is 3 if and only if R satisfies one of the following:*
   (a) *$R \cong R_1 \times R_2$ and $\mathfrak{m}_i$ is principal in R_i:*
      (i) *if $\eta_1 = 3$ and $\eta_2 = 2$, then $\mathfrak{m}_1$ and $\mathfrak{m}_1^2$ are the only non-trivial principal ideals in R_1 and $\mathfrak{m}_2$ is the only non-trivial principal ideal in R_2;*
      (ii) *if $\eta_1 = 2$ and $\eta_2 = 3$, then $\mathfrak{m}_1$ is the only non-trivial principal ideal in R_1 and $\mathfrak{m}_2$ and $\mathfrak{m}_2^2$ are the only non-trivial principal ideals in R_2.*
   (b) *$R \cong R_1 \times F_1$:*
      (i) *if $\mathfrak{m}_1 = \langle b_1, b_2\rangle$, then R_1 contains $\langle b_1\rangle$, $\langle b_2\rangle$, $\langle b_1b_2\rangle$, $\langle b_1 + b_2\rangle$ as the only non-trivial principal ideals;*
      (ii) *if $\mathfrak{m}_1 = \langle b_1\rangle$ is a principal ideal in R_1 with nilpotency $\eta = 5$, then $\mathfrak{m}$, $\mathfrak{m}^2$, $\mathfrak{m}^3$ and $\mathfrak{m}^4$ are the only non-trivial principal ideals of R_1.*

(2) *The book thickness of $\Gamma_r(R)$ is 4 if and only if R is isomorphic to $R_1 \times F_1$ and $\mathfrak{m}_1$ is principal ideal in R_1 with nilpotency 6.*

(3) *The book thickness of $\Gamma_r(R)$ is 5 if and only if R is isomorphic to $R_1 \times F_1 \times F_2$ and $\mathfrak{m}_1$ is the only nonzero proper principal ideal in R_1.*

Proof. Since planar reduced cozero-divisor graphs are two-page embeddable, we require at least three pages to embed toroidal reduced cozero-divisor graphs. Note that toroidal reduced cozero-divisor graphs are given in Theorems 1.12 and 1.13. Figures 3.4, 3.5, 3.6, and 3.7 exhibit a 3-book embedding of the respective rings. In the case of ring $R_1 \times F_1$ with nilpotency six, a 4-book embedding is given in Figure 3.8. For one more remaining case, a 5-book embedding is given in Figure 3.9. □

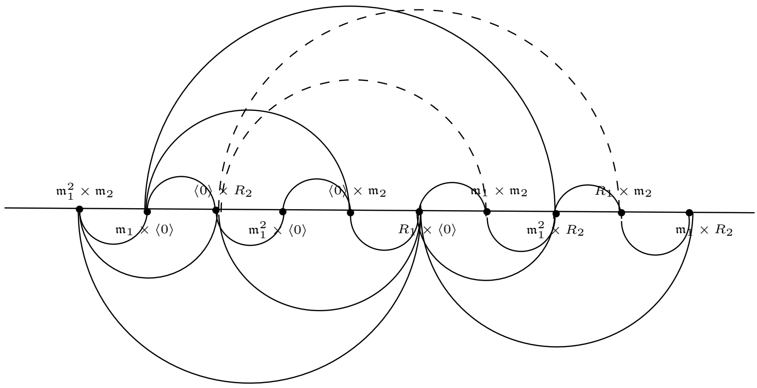


FIGURE 3.4. Three-page book embedding of $\Gamma_r(R_1 \times R_2)$ with $\eta_1 = 3$ and $\eta_2 = 2$

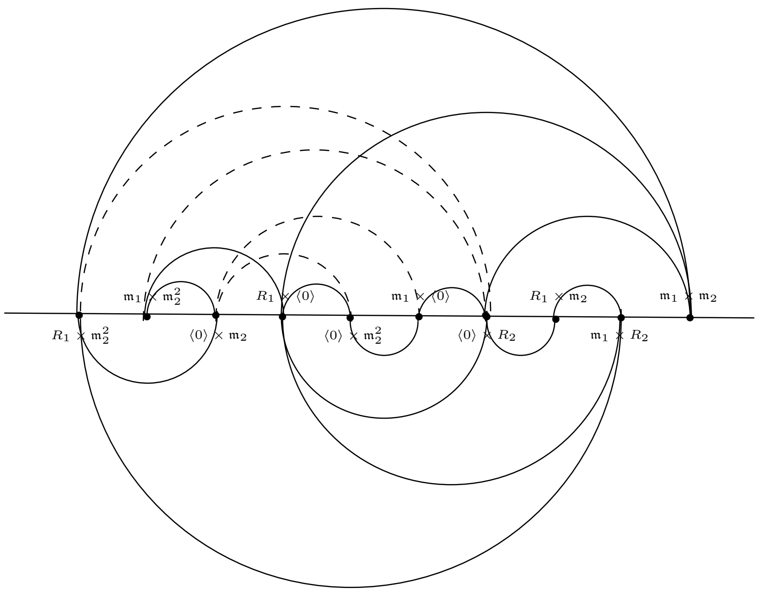


FIGURE 3.5. Three-page book embedding of $\Gamma_r(R_1 \times R_2)$ with $\eta_1 = 2$ and $\eta_2 = 3$

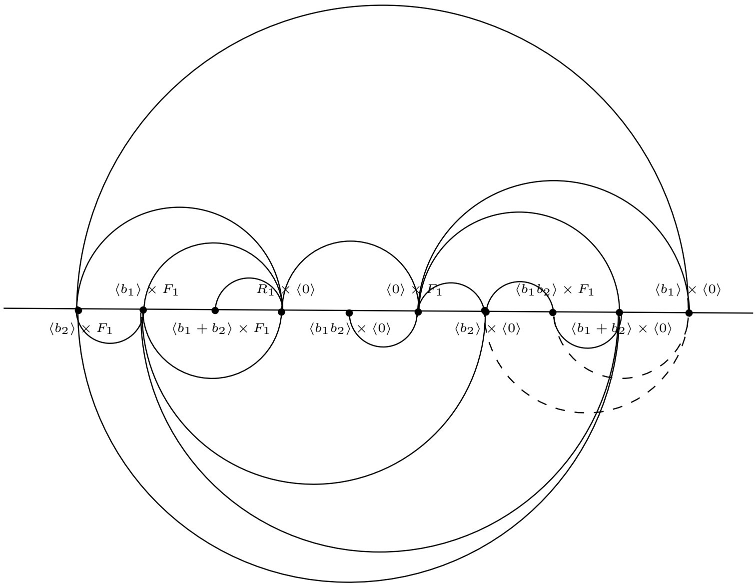


FIGURE 3.6. Three-page book embedding of $\Gamma_r(R_1 \times F_1)$ and $t = 2$

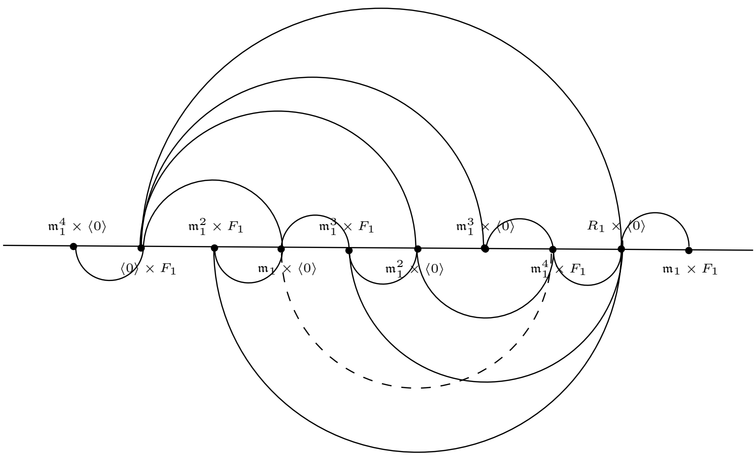


FIGURE 3.7. Three-page book embedding of $\Gamma_r(R_1 \times F_1)$ and $t = 1$, $\eta_1 = 5$

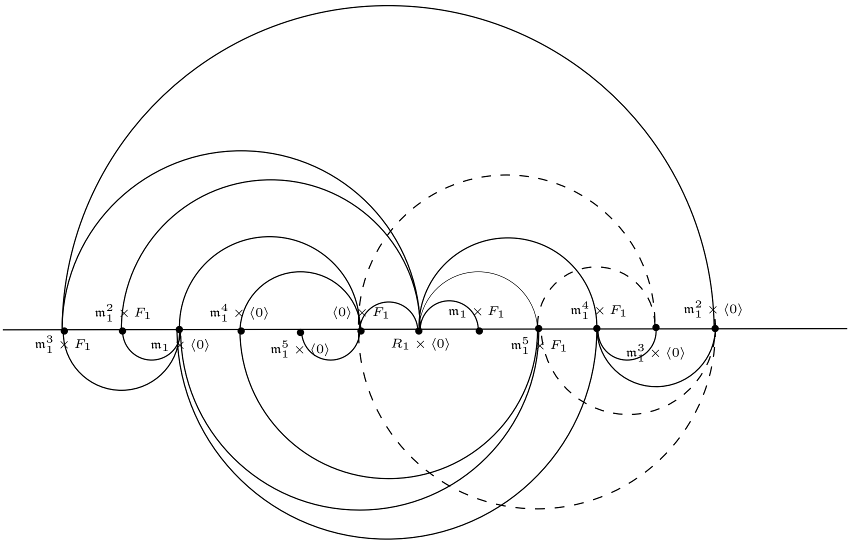


FIGURE 3.8. Four-page book embedding of $\Gamma_r(R_1 \times F_1)$ with $\eta_1 = 6$

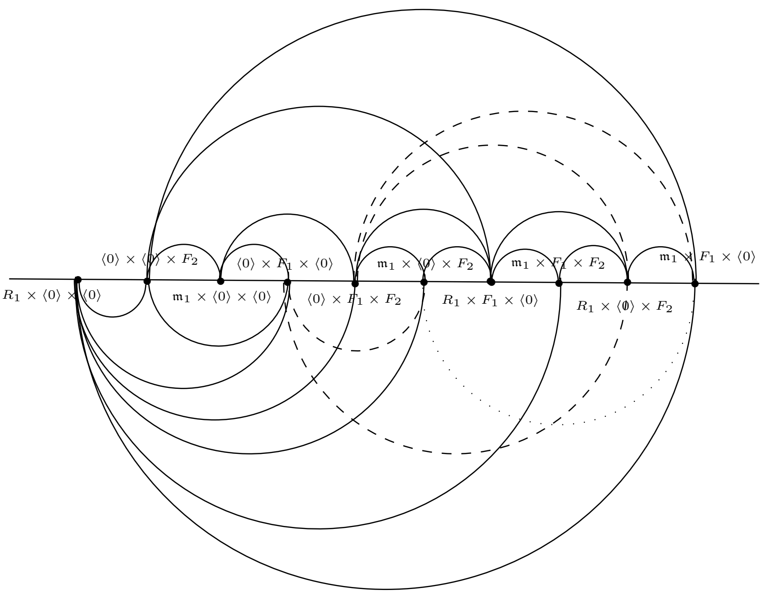


FIGURE 3.9. Five-page book embedding of $\Gamma_r(R_1 \times F_1 \times F_2)$

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